Local Tournaments with the minimum number of Hamiltonian cycles or cycles of length three

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Abstract

A digraph without loops, multiple arcs and directed cycles of length two is called a local tournament if the set of in-neighbors as well as the set of out-neighbors of every vertex induces a tournament.

In this paper we consider the following problem: Given a strongly connected local tournament D of order n and an integer $3 \leq r \leq n$. How many directed cycles of length r exist in D?

Bang-Jensen [1] showed in 1990 that every strongly connected local tournament has a directed Hamiltonian cycle, thus solving the case $r = n$. In 2009, Meierling and Volkmann [8] showed that a strongly connected local tournament D has at least $n - r + 1$ directed cycles of length r for $4 \leq r \leq n - 1$ unless it has a special structure.

In this paper, we investigate the case $r = 3$ and present a lower bound for the number of directed cycles of length three. Furthermore, we characterize the classes of local tournaments achieving equality in the bounds for $r = 3$ and $r = n$, respectively.

Keywords: Local Tournament; Number of Cycles

1 Terminology and Introduction

All digraphs mentioned here are finite without loops, multiple arcs and directed cycles of length two unless noted otherwise. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_r\}$ and let $H_1, H_2, \ldots . H_r$ be a collection of digraphs. Then $D[H_1, H_2, \ldots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and adding the arcs from every vertex of H_i to every vertex of H_j if $v_i v_j$ is an arc of D for all i and j satisfying $1 \leq i \neq j \leq r$.

If $xy \in E(D)$, then y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by $A \to B$. The *outset* $N^+(x)$ of a vertex x is the set of out-neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A, B)$ are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x) : x \in V(D)\}$ and $\min\{d^-(x) : x \in V(D)\}$, respectively. Furthermore, let $\delta(D)$ denote the minimum of $\delta^+(D)$ and $\delta^-(D)$. A digraph D is said to be r*-regular* if the indegree and outdegree of every vertex is equal to r.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \dots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \le i, j \le k$, denotes the subpath $x_ix_{i+1} \dots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then $x^+ = x^+_C$ denotes its successor on C and $x^- = x^-_C$ denotes its predecessor on C. The notations for paths are defined analogously.

A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. We say that a digraph D is k*-strongly connected* or just k*-strong* if D has at least $k + 1$ vertices and for any set S of vertices such that $|S| < k$, the subdigraph $D - S$ is strong. A set S of vertices is called a *separating set* if $D - S$ is not strong. We speak of a *minimal separating set* (*minimum separating set*) S of a digraph D if $D - S$ is not strong and there exists no separating set $S' \subseteq S$ with $|S'| < |S|$ (no separating set S'' with $|S''| < |S|$).

A digraph is *semicomplete* if for any two different vertices, there is either exactly one arc or a cycle of length two between them. A *tournament* is a semicomplete digraph without cycles of length two, e.g., an orientation of a complete undirected graph.

Throughout this paper all subscripts are taken modulo the corresponding number.

In 1966, Moon [9] proved the following result concerning the number of cycles of a specific length in tournaments.

Theorem 1.1 (Moon [9] 1966)**.** *Let T be a strong tournament on n vertices and let r be an integer such that $3 \le r \le n$. Then T has at least $n - r + 1$ cycles of length r for every $3 \le r \le n$.*

In 1975, Las Vergnas [7] characterized all strongly connected tournaments with minimum number of cycles of a given length $4 \leq r \leq n-1$.

Theorem 1.2 (Las Vergnas [7] 1975)**.** *Let T be a strong tournament on $n \geq 5$ vertices. Then T has at least $n-r+2$ cycles of length r for every $4 \leq r \leq n-1$ unless T is isomorphic to Q_n, where Q_n is the tournament of order n consisting of a path $x_1x_2 \ldots x_n$ and all arcs x_ix_j for $i > j+1$.*

The class of strongly connected tournaments with exactly one Hamiltonian cycle was characterized by Douglas [5] in 1970, and the class of strongly connected tournaments with exactly $n-2$ cycles of length three was characterized by Burzio and Demaria [4] in 1990.

In 1990, Bang-Jensen [1] defined *locally semicomplete digraphs* to be the family of digraphs where the in-neighborhood as well as the out-neighborhood of every vertex induces a semicomplete digraph. In transferring the general adjacency only to vertices that have a common out- or a common in-neighbor, locally semicomplete digraphs are an interesting generalization of semicomplete digraphs. An important subclass is the class of *local tournaments* which consists of locally semicomplete digraphs without cycles of length two. Note that therefore all results concerning locally semicomplete digraphs apply to local tournaments as well.

In his initial paper [1], Bang-Jensen showed, among other things, that a locally semicomplete digraph is strong if and only if it has a Hamiltonian cycle. Note that this result is a generalization of Moon's Theorem 1.1 for the case $r = n$.

In 2009, Meierling and Volkmann [8] transferred Moon's Theorem 1.1 to the class of local tournaments. In order to state their results we need the following definitions as well as an additional structural result.

Definition 1.3. *A digraph on n vertices is called a* round digraph *if its vertices $v_1, v_2, \ldots, v_n$ can be labelled such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-d^-(v_i)}\}$ for every i, where the subscripts are taken modulo n. We refer to $v_1, v_2, \ldots, v_n$ as a* round labeling *of D.*

Definition 1.4. *A locally semicomplete digraph D is* round decomposable *if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[H_1, H_2, \ldots, H_r]$, where each H_i is a strong semicomplete digraph of D. We call $R[H_1, H_2, \ldots, H_r]$ a* round decomposition *of D.*

Definition 1.5. *Let D be a strongly connected locally semicomplete digraph. The* quasi-girth *$g(D)$ of D is defined as follows: If D is round decomposable and it has the round decomposition $D = R[H_1, H_2, \ldots, H_r]$, then $g(D)$ is the length of a shortest cycle in R.*

We are now able to present Meierling's and Volkmann's results. Note that every strongly connected (local) tournament on less than five vertices is round decomposable.

Theorem 1.6 (Meierling & Volkmann [8] 2009)**.** *Let D be a strong local tournament on $n \geq 5$ vertices that is not round decomposable. Then D has at least $n - r + 1$ cycles of length r for every $4 \leq r \leq n$.*

Theorem 1.7 (Meierling & Volkmann [8] 2009)**.** *Let D be a strongly connected and round decomposable local tournament on $n \geq 5$ vertices with the round decomposition $R[D_1, D_2, \ldots, D_p]$, where $p \geq 3$. Let C_R be a shortest cycle of length $g(D)$ in R. Then*

(a) D has at least $n - r + 2$ cycles of length r for every $g(D) + 2 \leq r \leq n - 1$;

(b) D has at least $n - g(D)$ cycles of length $g(D) + 1 \leq n - 1$ with equality if and only if

- *(i) $|V(D_i)| = 1$ for every component $D_i \in V(C_R)$;*
- *(ii) if $D_i \in V(C_R)$, then $D - x$ is not strong for every vertex $x \in V(D_i)$;*
- *(iii) if $D_j \notin V(C_R)$, then $|V(D_j)| \leq g(D)$.*

They also proved the following result which is a generalization of Las Vergnas' Theorem 1.2.

Theorem 1.8 (Meierling & Volkmann [8] 2009)**.** *Let D be a strong local tournament on $n \geq 5$ vertices with at least $n + 2$ arcs. If D is neither isomorphic to Q_n nor to Q_5^* nor to a member of $\mathcal{R}^*$, then D has at least $n - r + 2$ cycles of length r for every $g(D) + 1 \leq r \leq n - 1$. (Here Q_n is defined as in Theorem 1.2, Q_5^* is defined by $Q_5 - x_4x_2$, and $\mathcal{R}^*$ is the class of local tournaments that fulfill conditions (i), (ii) and (iii) of Theorem 1.7 (b)).*

By presenting a class of strongly connected local tournaments on n vertices with $n - 3$ cycles of length three, Meierling and Volkmann [8] showed that Theorem 1.1 cannot be transferred directly to the class of local tournaments.

In this paper we show that every strongly connected local tournament on n vertices that is not round decomposable has at least $n - 3$ cycles of length three and characterize the local tournaments achieving equality in this bound. In addition, the class of strongly connected local tournaments with exactly one Hamiltonian cycle is elaborated.

2 Preliminary results

In his initial article [1] on local tournaments, Bang-Jensen introduced an important decomposition of locally semicomplete digraphs.

Theorem 2.1 (Bang-Jensen [1] 1990)**.** *Let D be a strongly connected locally semicomplete digraph and let S be a minimal separating set of D.*

(a) If A and B are two strong components of $D - S$, then either there is no arc between them or A dominates B or B dominates A;

(b) If A and B are two strong components of $D - S$ such that A dominates B, then $D[A]$ and $D[B]$ are semicomplete;

(c) The strong components of $D-S$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$, where $p \geq 2$, such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.

The unique sequence $D_1, D_2, \ldots, D_p$ is called the strong decomposition *of $D - S$. Furthermore, we call D_1 the* initial strong component *and D_p the* terminal strong component *of $D - S$.*

From the fact that every connected non-strong locally semicomplete digraph has a unique strong decomposition, Guo and Volkmann [6] found a further useful decomposition, which is formulated in the next theorem (see Theorem 3.3 in [2] for the version stated here).

Theorem 2.2 (Guo & Volkmann [6] 1994)**.** *Let D be a strongly connected locally semicomplete digraph and let S be a minimal separating set of D. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - S$. Then $D - S$ can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, \ldots, D'_r$ as follows:*

$$D'_1 = D_p, \lambda_1 = p, \lambda_{i+1} = \min\{j : N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

$$D'_{i+1} = D[V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1} \cup \ldots \cup V(D_{\lambda_i - 1}))].$$

The subdigraphs $D'_1, D'_2, \ldots, D'_r$ satisfy the properties below:

(a) D'_i consists of some strong components of D and is semicomplete for $i = 1, 2, \ldots, r$;

(b) D'_{i+1} dominates the initial strong component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \ldots, r-1$;

(c) if $r \geq 3$, then there is no arc between D'_i and D'_j for i and j satisfying $|j-i| \geq 2$.

The unique sequence $D'_1, D'_2, \ldots, D'_r$ is called the semicomplete decomposition *of $D - S$.*

Furthermore, Bang-Jensen, Guo, Gutin and Volkmann [2] showed that locally semicomplete digraphs which are not semicomplete have the following structure.

Theorem 2.3 (Bang-Jensen, Guo, Gutin & Volkmann [2] 1997)**.** *Let D be a strong locally semicomplete which is not semicomplete. Then D is not round decomposable if and only if the following conditions are satisfied:*

(a) There is a minimal separating set S of D such that $D-S$ is not semicomplete and for each such S the digraph $D[S]$ is semicomplete and the semicomplete decomposition of $D-S$ has exactly three components D'_1, D'_2 and D'_3;

(b) There are integers α, β, μ, ν with $\lambda_2 \le \alpha \le \beta \le p-1$ and $p+1 \le \mu \le \nu \le p+q$ such that

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset \text{ or}$$
$$N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D-S$ and $D[S]$, respectively, and D_{λ_2} is the initial strong component of D'_2.

By refining the proof of the above result, Meierling and Volkmann [8] showed the following result.

Theorem 2.4 (Meierling & Volkmann [8] 2009)**.** *Let D be a strong local tournament which is not a tournament and not round-decomposable. Let S be a minimal separating set of D that satisfies the conditions of Theorem 2.3. Then one of the following possibilities holds.*

(a) There is a vertex $s \in S$ and vertices $x_1, x_2 \in V(D'_2)$ with $x_1 \to s \to x_2$ and D has a Hamiltonian cycle C such that x_1 is the predecessor of x_2 on C;

(b) There is a vertex $x \in V(D'_2)$ and vertices $s_1, s_2 \in S$ with $s_1 \to x \to s_2$ and D has a Hamiltonian cycle C such that s_1 is the predecessor of s_2 on C.

Checking the proof of Theorem 2.4 in [8], one may observe the following fact which is needed in Section 4.

Remark 2.5. *In the situation of Theorem 2.4:*

- *If (a) holds, then the vertices s, x_1 and x_2 can be chosen such that $x_1 \in V(D_\alpha)$ and $x_2 \in V(D_\beta)$ with $|\alpha - \beta| \le 1$.*

- *If (b) holds, then the vertices s_1, s_2 and x can be chosen such that $s_1 \in V(D_\mu)$ and $s_2 \in V(D_\nu)$ with $|\mu - \nu| \le 1$.*

The following assertion, due to Bang-Jensen, Guo, Gutin and Volkmann, provides an important property concerning round decomposable, strong locally semicomplete digraphs.

Proposition 2.6 (Bang-Jensen, Guo, Gutin & Volkmann [2] 1997)**.** *Let* $R[H_1, H_2, \ldots, H_\alpha]$ *be a round decomposition of a strong locally semicomplete digraph. Then, for every minimal separating set* S*, there are two integers* i *and* $k \geq 0$ *such that* $S = V(H_i) \cup \ldots \cup V(H_{i+k})$*.*

3 Local Tournaments with exactly one Hamiltonian cycle

In this section, we characterize the class of local tournaments which are not tournaments and have exactly one Hamiltonian cycle.

Lemma 3.1. *Let* D *be a strongly connected local tournament,* S *a separating set of* D*,* $D_1, D_2, \ldots, D_p$ *the strong decomposition of* $D - S$ *and* $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ *the strong decomposition of* $D[S]$*, where* $p \geq 2$ *and* $q \geq 1$*. If* D *has a unique Hamiltonian cycle and* $D_{i-1} \rightarrow D_i \rightarrow D_{i+1}$*, then* $|V(D_i)| = 1$*.*

Proof. Assume that $|V(D_i)| \geq 3$. Let P be a Hamiltonian path of $D[V(D_{i+1}) \cup V(D_{i+2}) \ldots \cup V(D_{i-1})]$ that starts in y, x an arbitrary vertex of D_i, and C_i a Hamiltonian cycle of D_i. Then $PC_i[x, x^-]y$ and $PC_i[x^+, x]y$ are two Hamiltonian cycles of D. □

Corollary 3.2. *Let* D *be a strong local tournament with a unique Hamiltonian cycle.*

(i) If $D = R[D_1, D_2, \ldots, D_t]$ *is round decomposable, then* $|V(D_i)| = 1$ *for* $i = 1, 2, \ldots, t$*.*

(ii) If D *is not a tournament, there is a minimal separating set* S *of* D *such that* $D - S$ *satisfies Theorem 2.3. Then* $|V(D_i)| = 1$ *for* $i = 1, 2, \ldots, p + q$*.*

Proof. If $D = R[D_1, D_2, \ldots, D_t]$ is round decomposable, then $D_{i-1} \rightarrow D_i \rightarrow D_{i+1}$ for $i = 1, 2, \ldots, t$ by definition. Therefore (i) is true by Lemma 3.1.

If D is not a tournament, let S be a minimal separating set of D such that $D - S$ satisfies Theorem 2.3. Then $D_i \rightarrow D_{i+1}$ for $i = 1, 2, \ldots, p-1$ and $i = p+1, p+2, \ldots, p+q-1$. It remains to show that $D_p \rightarrow D_{p+1}$ and $D_{p+q} \rightarrow D_1$. Since D is strong, $N^+(D_{p+q}) \cap V(D_1) \neq \emptyset$ and $N^+(D_p) \cap V(D_{p+1}) \neq \emptyset$. If $D_{p+q} \not\rightarrow D_1$, there are vertices $x \in V(D_1)$ and $s \in V(D_{p+q})$ such that $x \rightarrow s$. Since S is a minimal separating set of D, s has an in-neighbor y in D_p. Using the local tournament property of D, we conclude that x and y are adjacent, a contradiction. We can analogously show that $D_p \rightarrow D_{p+1}$. The validity of (ii) follows by Lemma 3.1. □

We look now at local tournaments that are round decomposable.

Theorem 3.3. *Let $D = R[D_1, D_2, \ldots, D_t]$ be a k-strongly connected, round decomposable local tournament, where $t \geq 3$ and $k \geq 1$. Then D has a unique Hamiltonian cycle if and only if $|V(D_i)| = 1$ for $i = 1, 2, \ldots, t$ and*

(i) $k = 1$ or

(ii) $k = 2$, t is even and D is 2-regular.

Proof. Suppose that $V(D_i) = \{x_i\}$ for $i = 1, 2, \ldots, t$. Clearly, $C = x_1x_2 \ldots x_tx_1$ is a Hamiltonian cycle of D.

If $k = 1$, we may assume, without loss of generality, that $D - x_t$ is not strong. Furthermore, if C is not unique, D has a Hamiltonian cycle C' that contains an arc x_ix_j, where $j \geq i+2$. If $t \notin \{i, j\}$, all paths leading from x_j to x_i in D include x_t and thus, the vertex x_{i+1} is not included in C', a contradiction. Similarly, if $i = t$, C' does not include x_1, and if $j = t$, C' does not include x_{t-1}, again a contradiction.

If $k = 2$, t is even and D is 2-regular, it is easy to see that D is the second power of a t-cycle (see [3], Exercise 6.8). Hence $E(D)$ consists of the arcs x_ix_{i+1} and x_ix_{i+2}, where $i = 1, 2, \ldots, t$. If C is not unique, we may assume, without loss of generality, that D has a Hamiltonian cycle C' that contains x_2x_4. It follows that C' contains x_1x_3 and x_3x_5 and we can deduce inductively that C' uses all arcs x_ix_{i+2}, where $i = 1, 2, \ldots, t$. But then C' is not a cycle, a contradiction.

Now suppose that D has a unique Hamiltonian cycle. Then $|V(D_i)| = 1$ for $i = 1, 2, \ldots, t$ by Corollary 3.2. If $k = 1$, there is nothing to show. So let $k \geq 2$ and let S be a minimum separating set of D which is chosen such that $S = V(D_{p+1}) \cup V(D_{p+2}) \cup \ldots \cup V(D_{p+q})$ with $p + q = t$, and $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - S$ (this is possible by Proposition 2.6). Furthermore, let $D'_1, D'_2, \ldots, D'_r$ be the semicomplete decomposition of $D - S$ as defined in Theorem 2.2. Let $S = \{s_1, s_2, \ldots, s_q\}$ with $s_i \to s_j$ for $i < j$, and for $\ell = 1, 2, \ldots, r$ let $V(D'_\ell) = \{x^\ell_1, x^\ell_2, \ldots, x^\ell_{n_\ell}\}$, where $n_\ell = |V(D'_\ell)|$, with $x^\ell_i \to x^\ell_j$ for $i < j$. Note that D'_ℓ is a transitive subtournament of D. Let P_ℓ be the unique Hamiltonian path of D'_ℓ. Since D is round decomposable and $k \geq 2$, it follows that $x^1_1 \to S \to x^r_1$ and $x^\ell_{n_\ell} \to \{x^{\ell-1}_1, x^{\ell-1}_2\}$. The assumption that $k \geq 2$ yields $d^+(x^2_{n_2}) \geq 2$ and thus, $x^2_{n_2} \to s_1$, since D is round. We distinguish two cases.

Case 1: Suppose that $|V(D'_r)| \geq 2$. Then $s_q \to \{x^r_1, x^r_2\}$ and $s_{q-1} \to x^r_1$. It follows that

$$s_{q-1}x^r_1x^{r-1}_1 \ldots x^1_1s_qP_r[x^r_2, x^r_{n_r}]P_{r-1}[x^{r-1}_2, x^{r-1}_{n_{r-1}}] \ldots P_2[x^2_2, x^2_{n_2}]s_1s_2 \ldots s_{q-1}$$

is a second Hamiltonian cycle of D.

Case 2: Suppose that $|V(D'_r)| = 1$. Then $s_q \to \{x^r_1, x^{r-1}_1\}$ and $s_{q-1} \to x^r_1$.

Subcase 2.1: Suppose that $|V(D'_\ell)| \geq 3$ for an index $2 \leq \ell \leq r-1$. Then

$$s_{q-1}x_1^r P_{r-1}[x_2^{r-1}, x_{n_{r-1}}^{r-1}] \dots P_\ell[x_2^\ell, x_{n_\ell-1}^\ell] x_1^{\ell-1} \dots x_1^1 s_q$$
$$x_1^{r-1} \dots x_1^\ell x_{n_\ell}^\ell P_{\ell-1}[x_2^{\ell-1}, x_{n_{\ell-1}}^{\ell-1}] \dots P_2[x_2^2, x_{n_2}^2] s_1 s_2 \dots s_{q-1}$$

is an additional Hamiltonian cycle of D.

Subcase 2.2: Suppose that $|V(D'_\ell)| = 2$ for every index $2 \leq \ell \leq r-1$. Then we may assume that $k = q = 2$. Note that D has even order. Assume that there exists a vertex x in D such that $d^+(x) \geq 3$. We will show that D contains a second Hamiltonian cycle. Note that $x \notin \{x_2^3, x_2^4, \dots, x_2^{r-1}, x_1^1, x_1^r\}$.

If $d^+(s_1) \geq 3$, then $s_1 \to x_1^{r-1}$ and

$$s_1 x_1^{r-1} x_1^{r-2} \dots x_1^1 s_2 x_1^r x_2^{r-1} x_2^{r-2} \dots x_2^2 s_1$$

is a second Hamiltonian cycle in D.

If $d^+(s_2) \geq 3$, then $s_2 \to x_2^{r-1}$ and

$$s_2 x_2^{r-1} x_2^{r-2} \dots x_2^2 s_1 x_1^r x_1^{r-1} \dots x_1^1 s_2$$

is a second Hamiltonian cycle in D.

If $d^+(x_2^2) \geq 3$, then $N^+(x_2^2) = S \cup \{x_1^1\}$ and

$$s_1 x_1^r x_2^{r-1} x_2^{r-2} \dots x_2^2 s_2 x_1^{r-1} x_1^{r-2} \dots x_1^1 s_1$$

is an additional Hamiltonian cycle of D.

If $d^+(x_1^2) \geq 3$, then $x_1^2 \to s_1$ and

$$s_1 x_1^r x_2^{r-1} x_2^{r-2} \dots x_2^2 x_1^1 s_2 x_1^{r-1} x_1^{r-2} \dots x_1^2 s_1$$

is a second Hamiltonian cycle of D.

Finally, if $d^+(x_1^\ell) \geq 3$ for an index $3 \leq \ell \leq r-1$, then $N^+(x_1^\ell) = \{x_2^\ell, x_1^{\ell-1}, x_2^{\ell-1}\}$ and

$$s_1 x_1^r x_2^{r-1} x_2^{r-2} \dots x_2^\ell x_1^{\ell-1} x_1^{\ell-2} \dots x_1^1 s_2 x_1^{r-1} x_1^{r-2} \dots x_1^\ell x_2^{\ell-1} x_2^{\ell-2} \dots x_2^2 s_1$$

is an additional Hamiltonian cycle of D.

So $d^+(x) = 2$ for every vertex x of D. Since $d^-(x) \geq k = 2$, it follows that D is 2-regular and the proof is complete. □

We now turn our attention to strongly connected local tournaments that are not round decomposable.

Theorem 3.4. *Let D be a k-strongly connected local tournament that is not a tournament and not round decomposable. Let S be a separating set of D, $D_1, D_2, \dots, D_p$ be the semicomplete decomposition of $D - S$ as described in Theorem 2.3, and $D_{p+1}, D_{p+2}, D_{p+q}$ the strong decomposition of $D[S]$. Then D has a unique Hamiltonian cycle if and only if $k = 1$ and $|V(D_i)| = 1$ for $1 \leq i \leq p + q$.*

Proof. By Corollary 3.2, we assume that $|V(D_i)| = 1$ for $i = 1, 2, \dots, p + q$. If $k = 1$, it is easy to see that D has a unique Hamiltonian cycle. So let $k \geq 2$ and let C be a Hamiltonian cycle of D. Then, by Theorem 2.3 and Remark 2.5, there exist indices α and μ such that either

(i) $2 \leq \alpha \leq p - 2$, $p + 1 \leq \mu \leq p + q$, $V(D_\alpha) = \{v\}$, $V(D_{\alpha+1}) = \{w\}$ and $V(D_\mu) = \{u\}$ or

(ii) $2 \leq \alpha \leq p - 1$, $p + 1 \leq \mu \leq p + q - 1$, $V(D_\alpha) = \{u\}$, $V(D_\mu) = \{v\}$ and $V(D_{\mu+1}) = \{w\}$

and $v \to u \to w$. In both cases

$$C[u^+, v]uC[w, u^-]u^+$$

is a second Hamiltonian cycle in D. □

4 Local Tournaments with the minimum number of cycles of length three

By Theorem 1.7, the class of strongly connected, round decomposable local tournaments with the minimum number of cycles of a given length $r \geq 4$ is already characterized. The case $r = 3$ is treated in the next observation.

Observation 4.1. *Let $D = R[H_1, H_2, \dots, H_r]$ be a strongly connected, round decomposable local tournament.*

(a) If $g(D) \geq 4$, then D has no 3-cycles if and only if $|V(H_i)| = 1$ for $i = 1, 2, \dots, r$.

(b) If $g(D) = 3$, then D has at least one cycle of length three. It has exactly one 3-cycle if and only if $|V(H_i)| = 1$ for $i = 1, 2, \dots, r$ and R has exactly one 3-cycle.

In the remaining part of this section, we show that every strongly connected local tournament D that is not a tournament and not round decomposable has at least $n - 3$ cycles of length three and characterize the local tournaments achieving equality in this bound.

Theorem 4.2. *Let D be a strongly connected local tournament that is not a tournament. If D is not round decomposable, then D has at least $|V(D)|-3$ cycles of length three.*

Proof. Let S be a separating set of D such that $D-S$ fulfills the conclusions of Theorem 2.3. By Remark 2.5 there are indices α,β,μ,ν with $\lambda_2 \le \alpha \le \beta \le p-1$ and $p+1 \le \mu \le \nu \le p+q$ and vertices $x_1 \in V(D_\alpha)$, $x_2 \in V(D_\beta)$, $s_1 \in V(D_\mu)$ and $s_2 \in V(D_\nu)$ such that $x_1 \to s_1 \to x_2$ or $s_1 \to x_1 \to s_2$. By symmetry (reversing the direction of each arc of D), we may assume that $x \to s \to y$ with $x \in V(D_\alpha)$, $y \in V(D_\beta)$ and $s \in V(D_\mu)$. Recall that $\beta \in \{\alpha, \alpha+1\}$. Since $s \to D_1$ and $s \to y$, it follows that $D_1 \to y$. In addition, since $D_1 \to D_i$ for $i = 2, 3, \dots, D_{\lambda_2}$, we conclude that $D_i \to y$ for $i = 2, 3, \dots, D_{\lambda_2 - 1}$. Let x_1^3 be a vertex of D'_3 and let x_1^1 be a vertex of D'_1.

For every vertex $v \in V(D'_3)$ and every vertex $w \in V(D'_1)$, $svxs$ and $syws$, respectively, are cycles of length three in D.

If $s' \in S \setminus \{s\}$, then $s' \to s$ or $s \to s'$. Since D is a local tournament, in the first case s' is adjacent to x and in the latter case s' is adjacent to y. If $s' \to x$, then $s'xx_1^1s'$, if $x \to s'$, then $s'x_1^3xs'$, if $s' \to y$, then $s'yx_1^1s'$, and if $y \to s'$, then $s'x_1^3ys'$ is a cycle of length three in D.

If $z \in V(D'_2) \setminus \{x, y\}$ and $z \to y$ or $x \to z$, then z is adjacent to s, since D is a local tournament. If $s \to z$, then szx_1^1s is a cycle of length three in D. If $z \to s$ and $z \to y$, then sx_1^3zs is a cycle of length three in D. Finally, if $z \to s$ and $y \to z$, then $syzs$ is a cycle of length three in D.

If $z \in V(D'_2) \setminus \{x, y\}$ such that $y \to z \to x$, then $\alpha = \beta$ and $|V(D_\alpha)| \ge 3$. Since D_α is a strong subtournament of D, it contains at least $|V(D_\alpha)| - 2$ cycles of length three.

All in all, D contains at least

$$|V(D'_3)| + |V(D'_1)| + (|S|-1) + (|V(D'_2)| - 2) = |V(D)| - 3$$

cycles of length three. □

The local tournaments defined below show that Theorem 4.2 is sharp (see Figure 1).

Example 4.3. *We define four classes $\mathcal{T}_1^1, \mathcal{T}_2^1, \mathcal{T}_3^1, \mathcal{T}^2$ of strongly connected local tournaments that are not round decomposable.*

($\mathcal{T}_i^1$) *Each digraph D has the following properties. The vertex set of D consists of the four sets $S = \{s_1, s_2, \dots, s_q\}$, where $q \ge 1$, $A_1 = \{x_1^1\}$ and $A_k = \{x_1^k, x_2^k, \dots, x_{n_k}^k\}$ for $k = 2, 3$ such that $n_2 \ge 2$ and $n_3 \ge 1$. The arc set of D is defined as follows. Let $s_i \to s_j$ and $x_i^k \to x_j^k$ for $i < j$ and $k = 2, 3$, and let $A_2 \to x_1^1 \to S \to A_3$. Furthermore, let β be an integer with $2 \le \beta \le n_2$ and let $A_3 \to \{x_1^2, x_2^2, \dots, x_\beta^2\}$. If $\beta \ne 2$, we define $n_3 = 1$.*

$i = 1$: *Let $q \geq 2$ and $\beta = 2$. Additionally, D has the arcs $x_1^2 \to s_q \to x_2^2$, $A_3 \to \{x_3^2, x_4^2, \ldots, x_{n_2}^2\}$, $S \setminus \{s_q\} \to x_1^2$ and $s_q \to A_2 \setminus \{x_1^2\}$;*

$i = 2$: *Let $q = 1$. Additionally, D has the arcs $x_i^2 \to s_1 \to x_j^2$ for $i < \beta \leq j$ and $A_3 \to \{x_{\beta+1}^2, x_{\beta+2}^2, \ldots, x_{n_2}^2\}$;*

$i = 3$: *Let $n_3 = 1$. Additionally, D has the arcs $\{x_\beta^2, x_{\beta+1}^2, \ldots, x_{n_2}^2\} \to S \setminus \{s_1\}$ and $A_2 \setminus \{x_\beta^2\} \to s_1 \to x_\beta^2$.*

($\mathcal{T}^2$) *Each digraph D has the following properties. The vertex set of D consists of the four sets $S = \{s\}$, $A_1 = \{x_1^1\}$, $A_2 = \{x, y\} \cup Z' \cup Z$, where $Z' = \{z'_1, z'_2, \ldots, z'_r\}$ with $r \geq 1$ and $Z = \emptyset$ or $Z = \{z_1, z_2, \ldots, z_t\}$ with $t \geq 1$, and $A_3 = \{x_1^3, x_2^3, \ldots, x_{n_3}^3\}$ with $n_3 \geq 1$. The arc set of D is defined as follows. Let $x_i^3 \to x_j^3$, $z'_i \to z'_j$ and $z_i \to z_j$ for $i < j$, and let $A_2 \to x_1^1 \to S \to A_3 \to \{x, y\} \cup Z'$. Furthermore, let $x \to y$, $x \to s \to y$ and $y \to Z' \to x$. Moreover, let $\{x, y\} \cup Z' \to Z \to s$.*

We shall now show that these digraphs are the only strongly connected local tournaments that fulfill Theorem 4.2 with equality.

Theorem 4.4. *Let D be a strongly connected local tournament that is not a tournament. If D is not round decomposable, then D has exactly $|V(D)| - 3$ cycles of length three if and only if D is isomorphic to a member of $\mathcal{T}_i^1$ for $i = 1, 2, 3$ or $\mathcal{T}^2$.*

Proof. To see that every local tournament D that is isomorphic to a member of $\mathcal{T}_i^1$ for $i = 1, 2, 3$ or $\mathcal{T}^2$ has exactly $|V(D)| - 3$ cycles of length three, note that there is no 3-cycle in $D[S]$ or $D[A_j]$ for $j = 1, 3$. Furthermore, if D is a member of $\mathcal{T}_i^1$, there is no 3-cycle in $D[A_2]$.

If $D \in \mathcal{T}_1^1$, the only 3-cycles in D are $s_q v x_1^2 s_q$, where v is an arbitrary vertex of A_3, $s_q w x_1^1 s_q$, where $w \neq x_1^2$ is an arbitrary vertex of A_2, and $s' x_1^2 x_1^1 s'$, where s' in an arbitrary vertex of $S \setminus \{s_q\}$. Hence D has exactly $|A_3| + (|A_2| - 1) + (|S| - 1) = |V(D)| - 3$ cycles of length three.

If $D \in \mathcal{T}_2^1$, the only 3-cycles in D are $s_1 v w s_1$, where v is an arbitrary vertex of A_3 and $w = x_j^2$ with $1 \leq j < \beta$, and $s_1 w' x_1^1 s_1$, where $w' = x_j^2$ with $\beta \leq j \leq n_2$. So, if $\beta = 2$, then D has exactly $|A_3| + (|A_2| - 1) = |V(D)| - 3$ cycles of length three. If $\beta > 2$, then $|A_3| = 1$ by definition. It follows that D has exactly $(\beta - 1) + (|A_2| - (\beta - 1)) = n_2 = |V(D)| - 3$ cycles of length three.

If $D \in \mathcal{T}_3^1$, the only 3-cycles in D are $s_1 x_1^3 v s_1$, where $v = x_j^2$ with $1 \leq j < \beta$, $s_1 x_\beta^2 w s_1$, where $w = x_j^2$ with $\beta < j \leq n_2$, $s_1 x_\beta^2 x_1^1 s_1$ and $s' x_1^3 x_\beta^2 s'$, where s' is an arbitrary vertex of $S \setminus \{s_1\}$. Therefore, D has exactly $(\beta - 1) + (|A_2| - \beta) + 1 + (|S| - 1) = |A_2| + |S| - 1 = |V(D)| - 3$ cycles of length three.

If $D \in \mathcal{T}_2$, the only 3-cycles in D are $svxs$, where v is an arbitrary vertex of A_3, $syzs$, where z is an arbitrary vertex of Z, $syx_1^1 s$ and $xyz'x$, where z' is an

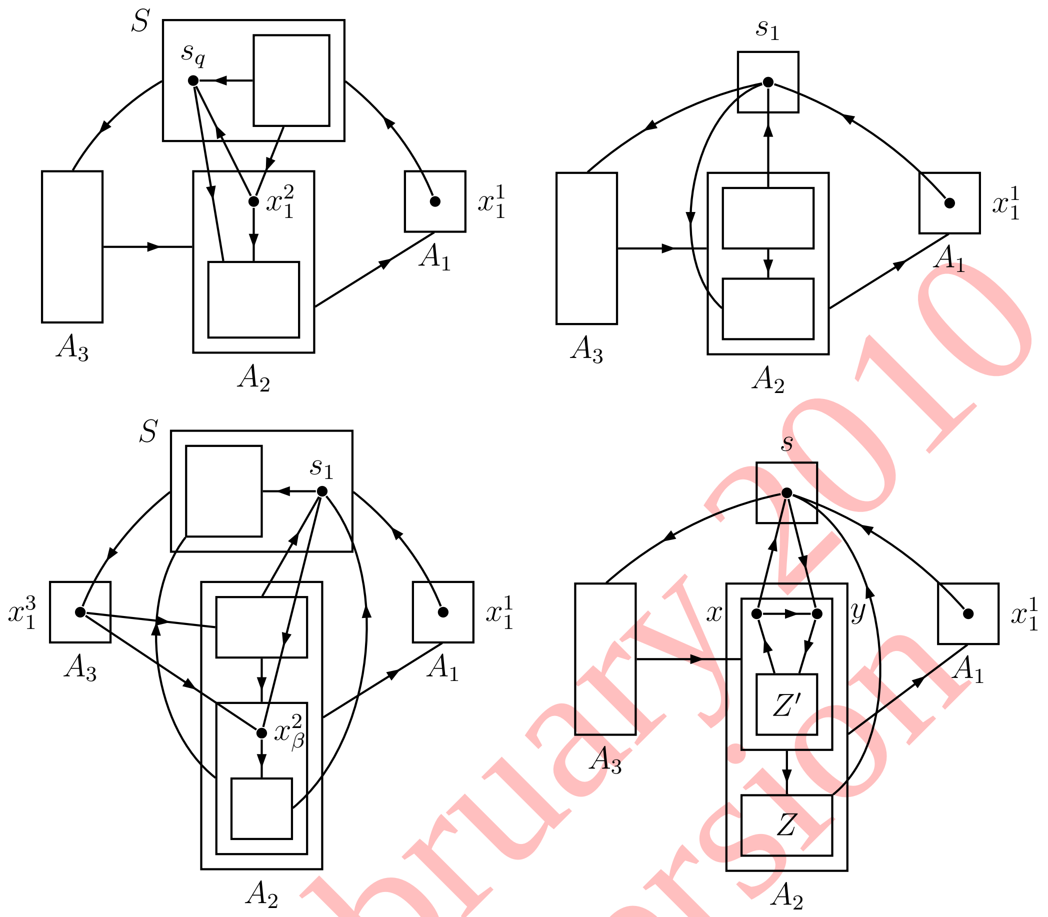

Figure 1: Local tournaments on n vertices with exactly $n-3$ cycles of length three: $\mathcal{T}_1^1$ (upper left), $\mathcal{T}_2^1$ (upper right), $\mathcal{T}_3^1$ (lower left), $\mathcal{T}^2$ (lower right).

arbitrary vertex of Z'. So, D has exactly $|A_3| + |Z| + 1 + |Z'| = |V(D)| - 3$ cycles of length three.

For the converse, assume that D is a local tournament with exactly $|V(D)|-3$ cycles of length three and that D is not round decomposable. Let S be a separating set of D such that $D - S$ fulfills the conclusions of Theorem 2.3. By Theorem 2.3 and Remark 2.5 there are indices α, β, μ, ν with $\lambda_2 \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ and vertices $x_1 \in V(D_\alpha)$, $x_2 \in V(D_\beta)$, $s_1 \in V(D_\mu)$ and $s_2 \in V(D_\nu)$ such that $x_1 \to s_1 \to x_2$ or $s_1 \to x_1 \to s_2$. By symmetry (reversing the direction of each arc of D), we may assume that $x \to s \to y$ with $x \in V(D_\alpha)$, $y \in V(D_\beta)$ and $s \in V(D_\mu)$. Note that $\beta \in \{\alpha, \alpha+1\}$.

If $\alpha \neq \beta$ and $|V(D_i)| \geq 3$ for any component or if $\alpha = \beta$ and $|V(D_i)| \geq 3$

for an index $i \neq \alpha$, the component D_i contains at least one 3-cycle that was not counted in Theorem 4.2 and thus, D contains at least $|V(D)| - 2$ cycles of length three, a contradiction. Therefore we assume $|V(D_i)| = 1$ for every index $1 \leq i \leq p + q$ if $\alpha \neq \beta$ and $|V(D_i)| = 1$ for every index $i \neq \alpha$ if $\alpha = \beta$. Let $S_1 = \{v \in S : v \to s\}$, $S_2 = \{w \in S : s \to w\}$, $Z_1 = \{v \in V(D'_2) : v \to \{x, y\}\}$, $Z_2 = \{w \in V(D'_2) : \{x, y\} \to w\}$ and $R = V(D'_2)\backslash(Z_1 \cup Z_2)$. Note that $R = \emptyset$ if and only if $\alpha \neq \beta$. Let $V(D'_3) = \{x^3_1, x^3_2, \dots, x^3_{n_3}\}$ and $V(D'_1) = \{x^1_1\}$. Throughout the remaining part of the proof, we call a 3-cycle that was not counted in Theorem 4.2 an *additional* 3*-cycle*.

Fact 1. *If $S_1 \neq \emptyset$, then $S_1 \to x$, and if $S_2 \neq \emptyset$, then $y \to S_2$.*

Proof. Assume that $s_1 \in S_1$. Then s_1 is adjacent to x. If $x \to s_1$, then s_1 and y are adjacent. Hence, either $s_1 y x^1_1 s_1$ or $s_1 x^3_1 y s_1$ is an additional 3-cycle in D, a contradiction. Therefore $s_1 \to x$. We can analogously show that $y \to S_2$. □

Fact 2. *If $\alpha = \beta$, then $S_1 = S_2 = \emptyset$ and there is no arc between s and R.*

Proof. Note that $R \neq \emptyset$. Let $r \in R$. If s and r are adjacent, then either $s r x^1_1 s$ or $s x^3_1 r s$ is an additional 3-cycle in D, a contradiction. So there is no arc between s and R.

If $s_1 \in S_1$, then $s_1 \to x$ by Fact 1. Hence s_1 is adjacent to r. Therefore either $s_1 r x^1_1 s_1$ or $s_1 x^3_1 r s_1$ is an additional 3-cycle in D, again a contradiction. We can analogously show that $S_2 = \emptyset$. □

Fact 3. *If $\alpha = \beta$, then $y \to R \to x$ and R induces a transitive tournament in D.*

Proof. Note that $V(D_\alpha) = \{x, y\} \cup R$ and thus, every vertex of R is adjacent to x and y. If there is a vertex $r \in R$ that dominates y or is dominated by x, then r is adjacent to s, a contradiction to Fact 2. If $D[R]$ is not transitive, then it contains an additional cycle of length three, again a contradiction. □

Fact 4. *If $Z_1 \neq \emptyset$, then $Z_1 \to s$ and $V(D'_3) = \{x^3_1\}$.*

Proof. Note that $Z_1 \cup \{s\} \to y$ and thus, s is adjacent to every vertex of Z_1. If $s \to z_1 \in Z_1$, then $s z_1 x s$ is an additional 3-cycle in D, a contradiction.

Let $z_1 \in Z_1$. If $v \neq x^3_1$ is a vertex of D'_3, then $s v z_1 s$ is an additional 3-cycle in D, again a contradiction. □

Fact 5. *If $\alpha = \beta$, then $Z_1 = \emptyset$.*

Proof. Note that $R \neq \emptyset$. Assume that $z_1 \in Z_1$. By Fact 4 and the definition of Z_1, we have $z_1 \to \{s\} \cup R$. It follows that s is adjacent to every vertex of R, a contradiction to Fact 2. □

Fact 6. *If $Z_2 \neq \emptyset$, then either $s \to Z_2$ or $Z_2 \to s$. Furthermore, if $Z_2 \to s$, then there is no arc leading from D'_3 to Z_2, and if $s \to Z_2$, then $D'_3 \to Z_2$.*

Proof. Note that $x \to Z_2 \cup \{s\}$ and thus, s is adjacent to every vertex of Z_2. If there are vertices $z, z' \in Z_2$ such that $z \to \{s, z'\}$ and $s \to z'$, it follows that $x_1^3 \to z$. Hence sx_1^3zs is an additional 3-cycle in D, a contradiction. If there are vertices $z, z' \in Z_2$ such that $szz's$ is a cycle, then D has at least $|V(D)| - 2$ cycles of length three, again a contradiction. So either $s \to Z_2$ or $Z_2 \to s$.

If $Z_2 \to s$ and there is a vertex $v \in V(D'_3)$ such that $v \to z \in Z_2$, then svz_2s is an additional 3-cycle in D, a contradiction.

If $s \to Z_2$, then $s \to \{u, z\}$ for every pair of vertices $u \in V(D'_3), z \in Z_2$. It follows that $D'_3 \to Z_2$. □

Fact 7. *If $\alpha = \beta$ and $Z_2 \neq \emptyset$, then $Z_2 \to s$.*

Proof. Note that $R \neq \emptyset$. By Fact 6, either $s \to Z_2$ or $Z_2 \to s$. But if $s \to z_2 \in Z_2$, then $R \cup \{s\} \to z_2$ and thus, s is adjacent to every vertex of R, a contradiction to Fact 2. □

Fact 8. *If $S_2, Z_2 \neq \emptyset$, then $Z_2 \to S_2 \cup \{s\}$.*

Proof. Note that all vertices of S_2 and Z_2 are adjacent, since $y \to S_2 \cup Z_2$ by Fact 1. If $s_2 \to z_2$, where $s_2 \in S_2$ and $z_2 \in Z_2$, then $s_2z_2x_1^1s_2$ is an additional 3-cycle in D, a contradiction. So $Z_2 \to S_2$.

If $s \to z \in Z_2$, it follows that $x_1^3 \to z$. But then $s_2x_1^3zs_2$, where $s_2 \in S_2$, is an additional 3-cycle in D, again a contradiction. □

Fact 9. *If $S_2 \neq \emptyset$, there is no arc between S_2 and $Z_1 \cup \{x\}$ and no arc between D'_3 and Z_2. Furthermore, $V(D'_3) = \{x_1^3\}$.*

Proof. If $S_2 \ni s_2 \to z \in Z_1 \cup \{x\}$, then $s_2zx_1^1s_2$ and if $Z_1 \cup \{x\} \ni z \to s_2 \in S_2$, then $s_2x_1^3zs_2$ is an additional 3-cycle in D, a contradiction. So there is no arc between S_2 and $Z_1 \cup \{x\}$.

If $x_i^3 \to z \in Z_2$, then $Z_2 \neq \emptyset$ and thus, $Z_2 \to S_2$ by Fact 8. It follows that $s_2x_i^3zs_2$ is an additional 3-cycle in D, again a contradiction.

Let $s_2 \in S_2$. If $v \neq x_1^3$ is a vertex in D'_3, then s_2vys_2 is an additional 3-cycle in D, the final contradiction. □

Fact 10. *If $S_1, Z_2 \neq \emptyset$, then $s \to Z_2$.*

Proof. Note that either $s \to Z_2$ or $Z_2 \to s$ by Fact 6. If $z_2 \to s$, where $z_2 \in Z_2$, let $s_1 \in S_1$ be an arbitrary vertex. Then s_1 and z_2 are adjacent, since $\{s_1, z_2\} \to s$. If $s_1 \to z_2$, then $s_1z_2x_1^1s_1$ and if $z_2 \to s_1$, then $s_1xz_2s_1$ is an additional 3-cycle in D, a contradiction. □

Fact 11. *If $S_1 \neq \emptyset$, then $Z_1 = \emptyset$, $S_1 \to x$, $S_2 = \emptyset$ and there is no arc between S_1 and $Z_2 \cup \{y\}$.*

Proof. Let $s_1 \in S_1$. If $z_1 \in Z_1$, then $z_1 \to s$ by Fact 4. Therefore z_1 and s_1 are adjacent, since $s_1 \to s$. If $s_1 \to z_1$, then $s_1 z_1 x_1^1 s_1$, and if $z_1 \to s_1$, then $s_1 x_1^3 z_1 s_1$ is an additional 3-cycle in D, a contradiction. So $Z_1 = \emptyset$.

Note that x is adjacent to every vertex of S_1, since $S_1 \cup \{x\} \to s$. If $s_1 \in S_1$ such that $x \to s_1$, then s_1 and y are adjacent, since $x \to y$. If $s_1 \to y$, then $s_1 y x_1^1 s_1$, and if $y \to s_1$, then $s_1 x_1^3 y s_1$, is an additional 3-cycle in D, again a contradiction. So $S_1 \to x$.

If $s_2 \in S_2$, then s_2 and x are adjacent, since $S_1 \to \{x, s_2\}$. If $s_2 \to x$, then $s_2 x s s_2$ and if $x \to s_2$, then $s_2 x_1^3 x s_2$ is an additional 3-cycle in D, again a contradiction. So $S_2 = \emptyset$.

If $s_1 \in S_1$ and $z \in Z_2 \cup \{y\}$ are adjacent, then $s_1 z x_1^1 s_1$ (if $s_1 \to z$) or $s_1 v z s_1$ (if $z \to s_1$) is an additional 3-cycle in D, the final contradiction. □

Using the above facts, we shall now show that D is isomorphic to a member of $\mathcal{T}_i^1$ for $i = 1, 2, 3$ or $\mathcal{T}^2$.

Case 1: Suppose that $\alpha = \beta$. By Facts 2 and 5, we know that $S_1 = S_2 = Z_1 = \emptyset$ and there is no arc between s and R. Using Facts 3 and 7, we conclude that R induces a transitive tournament in D, $y \to R \to x$ and that $Z_2 \to s$ if $Z_2 \neq \emptyset$. By Fact 6, it follows that D contains no arc leading from D_3' to Z_2 if $Z_2 \neq \emptyset$. Hence, D is isomorphic to a member of $\mathcal{T}^2$.

Case 2: Suppose that $\alpha \neq \beta$. We consider two cases depending on S_1.

Subcase 2.1: Suppose that $S_1 \neq \emptyset$. Then $Z_1 = S_2 = \emptyset$, $S_1 \to x$ and there is no arc between S_1 and $Z_2 \cup \{y\}$ by Fact 11. Furthermore, if $Z_2 \neq \emptyset$, then $s \to Z_2$ by Fact 10. Hence $D_3' \to D_2'$ and thus, D is isomorphic to a member of $\mathcal{T}_1^1$.

Subcase 2.2: Suppose that $S_1 = \emptyset$. Note that if $S_2 \neq \emptyset$, then $y \to S_2$, there is no arc between S_2 and $Z_1 \cup \{x\}$ and no arc leading from D_3' to Z_2 by Fact 9. Furthermore, $V(D_3') = \{x_1^3\}$. In addition, if $Z_2 \neq \emptyset$, then $Z_2 \to S_2$ by Fact 8. Hence, all arcs adjacent to at least one vertex of S_2 are determined. Furthermore, note that if $Z_1 \neq \emptyset$, then $Z_1 \to s$ and $V(D_3') = \{x_1^3\}$ by Fact 4. Therefore all arcs adjacent to at least one vertex of Z_1 are determined. It remains to consider the arcs between Z_2 and s and between Z_2 and D_3'. So assume that $Z_2 \neq \emptyset$. But then the arcs in question are determined by Facts 6 and 8.

Hence, if $S_2 \neq \emptyset$, then D is isomorphic to a member of $\mathcal{T}_3^1$. Furthermore, if $S_2 = \emptyset$, then the following holds. If $s \to Z_2 \neq \emptyset$, then D is isomorphic to a member of $\mathcal{T}_2^1$, and if $\emptyset \neq Z_2 \to s$, then D is isomorphic to a member of $\mathcal{T}_3^1$. Moreover, if $Z_2 = \emptyset$ and $Z_1 \neq \emptyset$, then D is isomorphic to a member of $\mathcal{T}_3^1$, and if $Z_1 = Z_2 = \emptyset$, then D is isomorphic to a member of $\mathcal{T}_2^1$.

Therefore, in all possible cases D is isomorphic to a member of $\mathcal{T}_i^1$ for $i = 2$ or $i = 3$. □

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