Solution of a conjecture of Tewes and Volkmann regarding extendable cycles in in-tournaments

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Abstract

A directed cycle C of a digraph D is extendable if there exists a directed cycle C' in D that contains all vertices of C and an additional one. In 1989, Hendry defined a digraph D to be cycle extendable if it contains a directed cycle and every non-Hamiltonian directed cycle of D is extendable. Furthermore, D is fully cycle extendable if it is cycle extendable and every vertex of D belongs to a directed cycle of length three. In 2001, Tewes and Volkmann extended these definitions in considering only directed cycles whose length exceed a certain bound $3 \leq k < n$: a digraph D is k-extendable if every directed cycle of length t, where $k \leq t < n$, is extendable. Moreover, D is called fully k-extendable if D is k-extendable and every vertex of D belongs to a directed cycle of length k.

An in-tournament is an oriented graph such that the in-neighborhood of every vertex induces a tournament. This class of digraphs which generalizes the class of tournaments was introduced by Bang-Jensen, Huang and Prisner in 1993. Tewes and Volkmann showed that every connected in-tournament D of order n with minimum degree $\delta \geq 1$ is $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable. Furthermore, if D is a strongly connected in-tournament of order n with minimum degree $\delta = 2$ or $\delta > \frac{8n-17}{31}$, then D is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable. In this paper we shall see that if $3 \leq \delta \leq \frac{8n-17}{31}$, every vertex of D belongs to a directed cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$ which means that D is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable. This confirms a conjecture of Tewes and Volkmann.

Keywords: In-tournaments; extendable cycles.

1 Terminology and introduction

All digraphs mentioned here are finite without loops, multiple arcs and cycles of length two. The notation will generally follow the notation of Bang-Jensen and Gutin in [3]. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

If $xy \in E(D)$, then y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \rightarrow y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by $A \rightarrow B$. Furthermore, $A \leadsto B$ denotes the fact that there is no arc leading from B to A and at least one arc is leading from A to B. In this case we also say that A *weakly dominates* B. The *out-neighborhood* $N^+(x)$ of a vertex x is the set of out-neighbors of x. More generally, for an arbitrary subdigraphs A of D, the outset $N^+(A)$ is defined as $\bigcup_{a \in A} N^+(a) - A$. The *in-neighborhoods* $N^-(x)$ and $N^-(A)$ are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x)|x \in V(D)\}$ and $\min\{d^-(x)|x \in V(D)\}$, respectively. Furthermore, the *minimum degree* $\delta(D)$ of D is the minimum of $\delta^+(D)$ and $\delta^-(D)$.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \ldots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \leq i, j \leq k$, denotes the subpath $x_ix_{i+1} \ldots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then $x^+ = x^+_C$ denotes its successor on C and $x^- = x^-_C$ denotes its predecessor on C. The notations for paths are defined analogously.

A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y.

An n-*tournament* is an orientation of a complete undirected graph of order n. A *local tournament* is a digraph where the inset as well as the outset of every vertex induces a tournament and an *in-tournament* is a digraph where the inset of every vertex induces a tournament.

Throughout this paper all subscripts are taken modulo the corresponding number.

The class of local tournaments was defined by Bang-Jensen [1] in 1990. In transferring the general adjacency only to vertices that have a common out- or a common in-neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning local tournaments, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length two. In particular, the Ph.D. theses of Guo [6] and Huang [8] handled this subject in detail.

In claiming adjacency only for vertices that have a common out-neighbor, Bang-Jensen, Huang and Prisner [4] introduced a further generalization of local tournaments, the class of *in-tournaments*. Some problems concerning in-tournaments have been studied by Bang-Jensen, Huang and Prisner in their initial article [4].

Later on, Tewes and Volkmann [13, 14], Volkmann [15], Tewes [11, 12], Peters and Volkmann [10] and Meierling and Volkmann [9] focused on the cycle and path structure of this class of digraphs. For more information concerning different generalizations of tournaments, the reader may be referred to the survey article or the comprehensive monograph on digraphs of Bang-Jensen and Gutin [2, 3].

A cycle C of a digraph D is *extendable* if there exists a directed cycle C' in D such that $V(C) \subseteq V(C')$ and $|V(C)|+1 = |V(C')|$. In 1989, Hendry [7] defined a digraph D to be *cycle extendable* if D contains a cycle and every non-Hamiltonian cycle of D is extendable. Furthermore, D is *fully cycle extendable* if D is cycle extendable and every vertex of D belongs to a cycle of length three. Tewes and Volkmann [14] extended these definitions in considering only cycles whose length exceed a certain bound k.

Definition 1.1 (Tewes & Volkmann [14] 2001)**.** *Let D be a digraph of order n and let $3 \leq k \leq n$ be an integer. Then D is* k-extendable *if every cycle of length t, where $k \leq t < n$, is extendable. Moreover D is called* fully k-extendable *if D is k-extendable and every vertex of D belongs to a cycle of length k.*

Remark 1.2. *The terms* fully 3-extendable *and* fully cycle extendable *describe the same property. Obviously, every fully k-extendable digraph D is vertex k-pancyclic, i.e. every vertex belongs to cycles of lengths $\ell = k, k+1, \ldots, |V(D)|$.*

Concerning cycle-extendability Volkmann and Tewes [14] proved the following result.

Theorem 1.3 (Tewes & Volkmann [13] 1998)**.** *Let D be a connected in-tournament of order n with minimum degree $\delta \geq 1$. Then D is $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

Like it was mentioned in [13], this result is best possible in the sense that there are even strong tournaments of order n and minimum degree δ that are fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable, but contain a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor - 1$ that is not extendable.

Using Theorem 1.3, Volkmann and Tewes [14] showed that the following two degree conditions are sufficient for a strong in-tournament to be fully k-extendable for large values of k.

Theorem 1.4 (Tewes & Volkmann [14] 2001)**.** *Let D be a strong in-tournament of order $n \geq 6$ with $\delta(D) \geq 2$. Then every vertex of D is contained in a cycle of length $n-3$.*

Theorem 1.5 (Tewes & Volkmann [14] 2001)**.** *Let D be a strong in-tournament of order n with $\max\{\delta^-(D), \delta^+(D)\} \geq 2$. Then every vertex of D is contained in a cycle of length $n-2$.*

The following results are immediate corollaries.

Corollary 1.6 (Tewes & Volkmann [14] 2001)**.** *Every strong in-tournament D of order n with $\delta(D) \geq 2$ is fully $(n-3)$-extendable.*

Corollary 1.7 (Tewes & Volkmann [14] 2001)**.** *Every strong in-tournament D of order n with $\max\{\delta^-(D), \delta^+(D)\} \geq 2$ is fully $(n-2)$-extendable.*

Corollary 1.8 (Tewes & Volkmann [14] 2001)**.** *Every strong in-tournament D of order n with $\max\{\delta^-(D), \delta^+(D)\} \geq 2$ contains two distinct vertices x_1, x_2 such that $D - x_i$ is strong for $i = 1, 2$. In particular, D is fully $(n-1)$-extendable.*

Tewes and Volkmann also proved that the following extension of Theorem 1.3 is valid for special values of δ.

Theorem 1.9 (Tewes & Volkmann [14] 2001)**.** *Every strong in-tournament D of order n with minimum degree $\delta = 2$ or $\delta > \frac{8n-17}{31}$ is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

For $\delta = 1$ we have $n - \lfloor \frac{4\delta+1}{3} \rfloor = n - 1$ and there are several examples, e.g. the cycle on n vertices, which illustrate that not every strongly connected in-tournament with minimum degree 1 is fully $(n-1)$-extendable. However, Tewes and Volkmann stated the following conjecture.

Conjecture 1.10 (Tewes & Volkmann [14] 2001)**.** *Let D be a strongly connected in-tournament of order n with minimum degree δ. If $3 \leq \delta \leq \frac{8n-17}{31}$, then D is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

In order to prove Conjecture 1.10, we shall show in Section 4 of this paper that every vertex of a strongly connected in-tournament D with $3 \leq \delta(D) \leq \frac{8n-17}{31}$ belongs to a cycle of length $n - \lfloor \frac{4\delta(D)+1}{3} \rfloor$.

2 Preliminary results

The following results play an important role in our investigation. The first one is a useful observation about the interaction of a cycle and a vertex which lies outside.

Theorem 2.1 (Bang-Jensen, Huang & Prisner [4] 1993)**.** *Let D be an in-tournament and let $C = u_1u_2 \ldots u_su_1$ be a cycle in D. If $|N^+(x) \cap V(C)| \geq 1$ for a vertex $x \notin V(C)$, then either $x \to C$ or $u_i \to x \to u_{i+1}$ for an integer $1 \leq i \leq s$.*

In 1993, Bang-Jensen, Huang and Prisner [4] generalized the well known result of Camion [5] that a tournament is strong if and only if it is Hamiltonian to in-tournaments.

Theorem 2.2 (Bang-Jensen, Huang & Prisner [4] 1993)**.** *An in-tournament is strong if and only if it has a Hamiltonian cycle.*

The following result shows that every cycle of length s that has a chord contains a vertex whose removal results in a cycle of length $s-1$. It will be frequently used in Section 3.

Lemma 2.3 (Tewes & Volkmann [14] 2001)**.** *Let D be an in-tournament and let $C = u_1u_2\dots u_su_1$ be a cycle in D such that $s \geq 4$. If $u_i \to u_s$ for an index $2 \leq i \leq s-2$, then either $D[V(C) - \{u_m\}]$ is strong for an index $2 \leq m \leq s-1$ or $u_{s-1} \to u_j$ for every index j with $1 \leq j \leq i$.*

3 Extending cycles and deleting vertices

In this section we investigate the cycle structure of strong in-tournaments. In particular we analyze how cycles can be transformed into other cycles by adding, deleting or exchanging vertices. The first lemma shows how we can transform a given cycle into a longer cycle.

Lemma 3.1. *Let D be a strong in-tournament of order n and let C be a cycle of length $s \geq \frac{n}{2}$ in D. If $|N^+(x) \cap V(C)| \geq 1$ for a vertex $x \notin V(C)$, then x belongs to a cycle of length $s+1$.*

Proof. Let $C = u_1u_2\dots u_su_1$ be a cycle of length $s = n-k-1$ in D, where $0 \leq k \leq \frac{n}{2}-1$, and let $x \notin V(C)$ be a vertex that has at least one out-neighbor on C. Then either $x \to C$ or there exists an index i such that $1 \leq i \leq s$ and $u_i \to x \to u_{i+1}$ by Theorem 2.1.

If $u_i \to x \to u_{i+1}$ for an index $1 \leq i \leq s$, the cycle C can be extended by x and thus, x belongs to a cycle of length $s+1$ in D.

So assume that $x \to C$. Since D is strong, there exists a path leading from C to x. Let P be a shortest such path and let, without loss of generality, $V(C) \cap V(P) = \{u_s\}$. Note that $r = |V(P)| \leq n-s+1 = k+2 \leq n/2+1$. Then

$$P[u_s, x]C[u_{r-1}, u_s]$$

is a cycle of length $r-1+s-(r-2) = s+1$ that includes x which completes the proof of this lemma. $\square$

The next result shows how we can transform a given cycle into a different cycle of the same length.

Lemma 3.2. *Let D be a strong in-tournament of order n and let C be a cycle of length $s \geq \frac{n}{2}$ in D. If $|N^+(x) \cap V(C)| \geq 2$ and $d^-(x) \geq 2$ for a vertex $x \notin V(C)$, then x belongs to a cycle of length s.*

Proof. Let $C = u_1u_2 \dots u_su_1$ be a cycle of length $s = n-k$ in D, where $0 \le k \le \frac{n}{2}$, and let $x \notin V(C)$ be a vertex such that $|N^+(x) \cap V(C)| \ge 2$ and $d^-(x) \ge 2$.

If $x \to C$, there exists a path leading from C to x, since D is strong. Let P be a shortest such path and let, without loss of generality, $V(C) \cap V(P) = \{u_s\}$. Since $d^-(x) \ge 2$, it follows that $r = |V(P)| \le n - s = k \le n/2$. Then

$$P[u_s, x]C[u_r, u_s]$$

is a cycle of length $r - 1 + s - (r-1) = s$ that includes x.

So assume that $x \not\to C$. In view of Theorem 2.1 there exists an index i such that $u_i \to x \to u_{i+1}$, say $i = s$. Since $|N^+(x) \cap V(C)| \ge 2$, the vertex x has a out-neighbor on C besides u_1. If $x \to u_2$,

$$xC[u_2, u_s]x$$

is a cycle of length s through x. Hence assume that $x \not\to u_2$. Using Theorem 2.1, it follows that there exists an index $2 \le j \le s-2$ such that $u_j \to x \to u_{j+1}$. Then u_s and u_j are adjacent, say $u_j \to u_s$. In view of Lemma 2.3, it follows that $D[V(C) - \{u_m\}]$ is strong for an index $2 \le m \le s$ and thus, according to Theorem 2.2, Hamiltonian. Let C' be a Hamiltonian cycle of $D[V(C) - \{u_m\}]$ and note that $|N^+(x) \cap V(C')| \ge 1$ and $|N^-(x) \cap V(C')| \ge 1$. Applying Theorem 2.1 to x and C', we find a cycle of length s through x which completes the proof of this lemma. □

The final result in this section shows how we can transform a given cycle into a shorter cycle.

Lemma 3.3. *Let D be a strong in-tournament of order n with minimum degree $\delta \ge 2$ and let C be a cycle of length $s = n - k + 1 \ge \frac{n}{2} + 1$ in D, where $1 \le k < 2\delta - 1$. If C has a chord and $|N^+(x) \cap V(C)| \ge 1$ for a vertex $x \notin V(C)$, then x belongs to a cycle of length $s - 1$.*

Proof. Let $C = u_1u_2 \dots u_su_1$ be a cycle of length $s = n - k + 1$ in D and let $x \notin V(C)$ be a vertex that has a out-neighbor on C. Let, without loss of generality, $u_i \to u_s$ for an index i with $2 \le i \le s-2$. Applying Lemma 2.3, we conclude that $D[V(C) - \{u_m\}]$ is strong for an index m with $2 \le m \le s$. It follows that there exists a cycle C' of length $s-1$ such that $V(C') \subseteq V(C)$. Let $C' = v_1v_2 \dots v_rv_1$ be such a cycle of length $r = s-1$ such that the distance $d(x, C')$ from x to C' is minimal. Then $d(x, C') \le 2$. If $|N^+(x) \cap V(C')| \ge 2$, the vertex x belongs to a cycle of length $s-1$ in view of Lemma 3.2.

Thus, assume for the remaining part of the proof that $|N^+(x) \cap V(C')| \le 1$. We consider two cases depending on the number of out-neighbors of x on C'.

Case 1. Suppose that $|N^+(x) \cap V(C')| = 1$. We may assume, without loss of generality, that $v_r \to x \to v_1$ and $(C' - v_1) \rightsquigarrow x$.

If $v_i \to v_r$ for an index $2 \le i \le r-2$, the digraph $D[V(C') - \{v_t\}]$ is strong for an index t with $2 \le t \le r$ in view of Lemma 2.3. Since $x \to v_1 \ne v_t$, it follows that D has a cycle C^* of length $s-2$ such that $V(C^*) \subseteq V(C')$ and $|N^+(x) \cap V(C^*)| = 1$. By Theorem 2.1 we conclude that x belongs to a cycle of length $s-1$.

So assume that v_{r-1} is the only in-neighbor of v_r on C'. Then

$$N^+(x) - \{v_1\} \subseteq V(D) - V(C') - \{x\} \text{ and}$$
$$N^-(v_r) - \{v_{r-1}\} \subseteq V(D) - V(C') - \{x\}.$$

Furthermore, note that

$$\begin{aligned} |V(D) - V(C') - \{x\}| &= k - 1 < 2\delta - 2, \\ |N^+(x) - \{v_1\}| &\ge \delta - 1 \text{ and} \\ |N^-(v_r) - \{v_{r-1}\}| &\ge \delta - 1. \end{aligned}$$

Therefore

$$|N^+(x) - \{v_1\}| + |N^-(v_r) - \{v_{r-1}\}| > |V(D) - V(C') - \{x\}|$$

and thus, there exists a vertex $z \notin V(C')$ such that $x \to z \to v_r$. In addition, note that

$$\begin{aligned} |N^-(x) \cap (V(D) - V(C'))| &\le |V(D) - V(C') - \{x\}| - |N^+(x) \cap (V(D) - V(C'))| \\ &\le (k-1) - (\delta - 1) \\ &= k - \delta. \end{aligned}$$

It follows that

$$|N^-(x) \cap V(C')| \ge \delta - |N^-(x) \cap (V(D) - V(C'))| \ge \delta - (k - \delta) = 2\delta - k > 1.$$

We consider three subcases.

Subcase 1.1. Suppose that $v_{r-1} \to x$. Then

$$xC'[v_1, v_{r-1}]x$$

is a cycle of length $s-1$ through x.

Subcase 1.2. Suppose that $v_{r-2} \to x$. Then we consider the cycle

$$C^* = xC'[v_1, v_{r-2}]x$$

of length $r-1=s-2$. Note that v_r does not have a in-neighbor on C^*, since v_{r-1} is the only in-neighbor of v_r on C'. It follows by Theorem 2.1 that $v_r \to C^*$. Hence the digraph D contains the cycle

$$xzv_rC'[v_2, v_{r-2}]x$$

of length $s-1$ through x.

Subcase 1.3. Suppose that $v_i \to x$ for an index i with $2 \le i \le r-3$. In this case we consider the cycle

$$C^* = xzC'[v_r, v_i]x$$

of length $i+3 \le r$ through x. Note that none of the vertices v_j, where $i < j < r$, dominates C^*, since, according to our assumption, v_{r-1} is the only in-neighbor of v_r on C' and $v_{r-1} \not\to x$. Using Theorem 2.1, it follows that we can insert the vertices $v_{r-1}, v_{r-2}, \ldots, v_{i+3}$ in C^* resulting in a cycle of length $s-1$ through x.

Case 2. Suppose that $|N^+(x) \cap (C')| = 0$. By choice of C' we obtain that $d(x, C') = 2$. Let $P = xyz$ be a path leading from x to C'. Then $|N^+(y) \cap V(C')| \ge 1$ and following Case 1 we conclude that there exists a cycle of length $s-1$ in D that includes y, a contradiction to the choice of C' which completes the proof of this lemma. □

4 Solution of the conjecture

Using the results of Section 3, we show in this section that every vertex of a strongly connected in-tournament of order n with minimum degree δ such that $3 \le \delta \le \frac{3n-2}{8}$ belongs to a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$. Since $\frac{8n-17}{31} \le \frac{3n-2}{8}$ for every positive integer n, this result in combination with Theorem 1.3 solves Conjecture 1.10 in positive.

The first result will be needed in the proof of Theorem 4.2.

Lemma 4.1. *Let D be an in-tournament of order n and let $C = u_1u_2\ldots u_su_1$ be a cycle of length $s > \frac{n}{2}$ such that $d^+(x) \ge 3$ for every vertex x of C. Then C has a chord, i.e. there exists a pair i, j of indices such that $|i-j| \ge 2$ and $u_i \to u_j$.*

Proof. Since $d^+(x) > 1$ for every vertex $x \in V(C)$, the proposition is true for $s = n$ by Theorem 2.2.

Now let $C = u_1u_2\ldots u_su_1$ be a cycle of length $s < n$ in D and assume that C does not have a chord. It follows that every external vertex $y \notin V(C)$ has at most two in-neighbors on C. In addition, since $d^+(x) \ge 3$ for every vertex $x \in V(C)$, every vertex of C dominates at least two external vertices. Hence

$$2s \le |\{xy \in E(D) \mid x \in V(C), y \notin V(C)\}| \le 2(n-s)$$

which leads to $2s \le n$, a contradiction. □

We now show that every vertex of a strong in-tournament belongs to cycles of large length.

Theorem 4.2. *Let D be a strong in-tournament of order n with minimum outdegree $\delta^+ \geq 3$. If $0 \leq t < 2\delta(D) - 1$ and $t \leq \frac{n}{2}$, then every vertex of D belongs to a cycle of length $n - t$.*

Proof. If $\delta^-(D) = 1$, then $\delta(D) = 1$ which implies that $t = 0$. Hence the proposition is true by Theorem 2.2. So assume that $\delta^-(D) \geq 2$ for the remainder of this proof.

Firstly we shall show that there exists a cycle of length $n - t$ in D. Note that, in view of Theorems 2.2, 1.8 and 1.5, every vertex of D belongs to cycles of length n, $n-1$ and $n-2$. So assume that $t \geq 3$ and let C be a shortest cycle in D such that $|V(C)| \geq n - t$. Let $|V(C)| = n - k + 1$ be the length of C and assume that $3 \leq k \leq t$. Since D is strong, there exists a vertex $x \notin V(C)$ such that $|N^+(x) \cap V(C)| \geq 1$. Note that C has a chord by Lemma 4.1. In view of Lemma 3.3 the vertex x belongs to a cycle of length $n - k$ in D, a contradiction to the choice of C. Hence $k = t + 1$, i.e. D contains a cycle of length $n - t$.

We shall show next that every vertex of D belongs to a cycle of length $n - t$ in D. Assume that D has a vertex x that does not belong to a cycle of length $n - t$. Let $C = u_1u_2 \ldots u_su_1$ be a cycle of length $n - t$ such that x has minimal distance to C and let $P = x_0x_1 \ldots x_r$ be a shortest path leading from $x = x_0$ to C. Then $x_{r-1} \notin V(C)$ and $N^+(x_{r-1}) \cap V(C) \neq \emptyset$.

If $|N^+(x_{r-1}) \cap V(C)| \geq 2$, we conclude by Lemma 3.2 that x_{r-1} belongs to a cycle of length $n-t$, a contradiction to the choice of C. So assume that $|N^+(x_{r-1}) \cap V(C)| = 1$. We may assume, without loss of generality, that $u_s \to x_{r-1} \to u_1$. Note that C has a chord by Lemma 4.1. This implies, in view of Lemma 2.3, that $D[V(C) - \{u_m\}]$ is strong for an index $1 \leq m \leq s$.

If $D[V(C) - \{u_m\}]$ is strong for an index $m \neq 1$, let C' be a Hamiltonian cycle of $D[V(C) - \{u_m\}]$. Note that $|N^+(x_{r-1}) \cap V(C')| = 1$. In view of Theorem 2.1, it follows that x_{r-1} belongs to a cycle of length $|V(C')| + 1 = |V(C)|$, a contradiction to the choice of C.

So assume that $m = 1$ and that $D[V(C) - \{u_i\}]$ is not strong for every index $i > 1$. Then the only in-neighbor of u_s on C is u_{s-1} by Lemma 2.3. Furthermore, recall that the only out-neighbor of x_{r-1} on C is u_1. Let $C' = v_2v_3 \ldots v_sv_2$ be a Hamiltonian cycle of $D[V(C) - \{u_1\}]$ such that $v_s = u_s$. Since $|V(D) - V(C) - \{x_{r-1}\}| = t - 1$, it follows that

$$
\begin{aligned}
|N^+(x_{r-1}) \cap (V(D) - V(C) - \{x_{r-1}\})| + |N^-(v_s) \cap (V(D) - V(C) - \{x_{r-1}\})| \\
\geq 2\delta - 2 > t - 1 \\
= |V(D) - V(C) - \{x_{r-1}\}|
\end{aligned}
$$

and thus, there exists a vertex $z \notin V(C)$ such that $x_{r-1} \to z \to v_s$. Again, recall that x_{r-1} does not have any out-neighbors on C'. Thus, we conclude that

$$\begin{aligned}|N^-(x_{r-1}) \cap V(C')| &= |N^-(x_{r-1})| - |N^-(x_{r-1}) \cap (V(D) - V(C'))| \\ &\geq \delta - (t - \delta) = 2\delta - t > 1.\end{aligned}$$

Hence x_{r-1} has a in-neighbor on C' besides v_s. We consider three cases depending on the in-neighborhood of x_{r-1} on C'.

Case 1. Suppose that $v_{s-2} \to x_{r-1}$. Then

$$x_{r-1} z C'[v_s, v_{s-2}] x_{r-1}$$

is a cycle of length $n - t$ through x_{r-1}, a contradiction.

Case 2. Suppose that $v_{s-2} \not\to x_{r-1}$ and that there exists a vertex $v \neq v_{s-1}$ in C' that dominates x_{r-1}. Let $2 \leq t \leq s - 3$ be the greatest index such that $v_t \to x_{r-1}$. We consider the cycle

$$C^* = x_{r-1} z C'[v_s, v_t] x_{r-1}.$$

If $v_{s-1} \not\to C^*$, the cycle C^* can be extended by v_{s-1}. Since $v_{s-2} \not\to x_{r-1}$ and because of the choice of t none of the vertices $v_{s-2}, v_{s-3}, \ldots, v_{t+2}$ dominates x_{r-1}. It follows that we can insert these vertices one after the other in C^*, leading to a cycle of length $n - t$ that includes x_{r-1}.

So assume that $v_{s-1} \to C^*$. Then $v_2 v_3 \ldots v_{s-1} v_2$ is a cycle in D and u_1 has a out-neighbor on this cycle. If u_1 can be inserted in $v_2 v_3 \ldots v_{s-1} v_2$, the digraph $D[V(C) - \{u_s\}]$ is strong, a contradiction. Hence $u_1 \to \{v_2, v_3, \ldots, v_{s-1}\}$. But then

$$x_{r-1} u_1 C'[v_2, v_{s-1}] x_{r-1}$$

is a cycle of length $n - t$ through x_{r-1}, again a contradiction.

Case 3. Suppose that $N^-(x_{r-1}) \cap V(C') = \{v_{s-1}, v_s\}$. Note that u_1 has a out-neighbor on C', say $u_1 \to v_q$, where $2 \leq q \leq s - 1$. We consider the cycle

$$C^* = x_{r-1} u_1 C'[v_q, v_{s-1}] x_{r-1}.$$

If $q = 2$, the cycle C^* is a cycle of length $n - t$ through x_{r-1}. So assume that $q \geq 3$. Due to our assumption none of the vertices $v_{q-1}, v_{q-2}, \ldots, v_2$ dominates C^*. Therefore we can insert these vertices one after the other in C^*, leading to a cycle of length $n - t$ that includes x_{r-1}. This final contradiction completes the proof of Case 3 and of this theorem. □

Since the two requirements on $t = \lfloor \frac{4\delta+1}{3} \rfloor$ are fulfilled if $\delta \leq \frac{3n-2}{8}$, the above theorem immediately yields the following result.

Corollary 4.3. *Let D be a strong in-tournament of order n with minimum degree δ such that $3 \leq \delta \leq \frac{3n-2}{8}$. Then every vertex of D belongs to a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$.*

Since $\frac{8n-17}{31} \leq \frac{3n-2}{8}$ for every positive integer n, Theorems 1.3 and Corollary 4.3 solve Conjecture 1.10 in positive.

Corollary 4.4. *Let D be a strong in-tournament of order n such that $3 \leq \delta(D) \leq \frac{8n-17}{31}$. Then D is fully $(n - \lfloor \frac{4\delta(D)+1}{3} \rfloor)$-extendable.*

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