Solution of a conjecture of Volkmann on longest paths through an arc in strongly connected in-tournaments

Dirk Meierling

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
e-mail: meierling@math2.rwth-aachen.de

Abstract

An in-tournament is an oriented graph such that the negative neighborhood of every vertex induces a tournament. Let $m = 4$ or $m = 5$ and let D be a strongly connected in-tournament of order $n \geq 2m - 2$ such that each arc belongs to a directed path of order at least m. In 2000, Volkmann showed that if D contains an arc e such that the longest directed path through e consists of exactly m vertices, then e is the only arc of D with that property. In this paper we shall see that this proposition is true for $m \geq 4$, thereby validating a conjecture of Volkmann. Furthermore, we prove that if we ease the restrictions on the order of D to $n \geq 2m - 3$, the in-tournament D in question has at most two such arcs. In doing so, we also give a characterization of the in-tournaments with exactly two such arcs.

1 Terminology and introduction

All digraphs mentioned here are finite without loops, multiple arcs and cycles of length two, unless noted otherwise. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

If $xy \in E(D)$, the vertex y is a *positive neighbor* of x and x is a *negative neighbor* of y, and we also say that x *dominates* y, denoted by $x \to y$. If A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, we say that A *dominates* B, denoted by $A \to B$. Furthermore, $A \leadsto B$ denotes the fact that there is no arc leading from B to A and at least one arc leading from A to B. In this case we also say that A *weakly dominates* B. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The insets $N^-(x)$ and $N^-(A, B)$ are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. A cycle or path of order m is an m*-cycle* or an m*-path*, respectively. If C is a cycle, let $C[x_i, x_j]$, where $1 \leq i, j \leq k$, denote the subpath $x_i x_{i+1} \ldots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then x_C^+ denotes the successor of x on C and x_C^- denotes the predecessor of x on C. If no confusion arises, x^+ and x^- will be used to denote x_C^+ and x_C^-. The notations for paths are defined analogously. The following definition plays an important role in this paper.

Definition 1.1. *Let D be a digraph. If the longest path through an arc e of D consists of m vertices, then we call e an m*-path arc *of D.*

We speak of a *connected digraph* if the underlying graph is connected. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of D is a maximal induced strong subdigraph of D. A digraph D is k*-connected* if for any set S of at most $k-1$ vertices the subdigraph $D - S$ is strong. If D is a strong digraph and S is a subset of $V(D)$ such that $D - S$ is not strong, we say that S is a *separating set*. A separating set S is called *minimal separating set* (*minimum separating set*) if there exists no separating set U such that $U \subsetneq S$ and $U \neq S$ ($|U| < |S|$).

An *in-tournament* is an oriented graph with the property that the inset of every vertex induces a tournament, i.e., every pair of distinct vertices that have a common positive neighbor are adjacent. A *local tournament* is an oriented graph such that the inset as well as the outset of every vertex induces a tournament.

Throughout this paper all subscripts are taken modulo the corresponding number.

Local tournaments were introduced by Bang-Jensen [1] in 1990. In transferring the general adjacency only to vertices that have a common positive or negative neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning this class of digraphs, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length two. In particular the Ph.D. theses of Guo [4] and Huang [6] handled this subject in detail. In 1993, Bang-Jensen, Huang and Prisner [3] introduced the family of in-tournaments as a further generalization of local tournaments. Since then, the properties of in-tournaments, especially the cycle and path structure, have been investigated by Tewes [9, 10, 11], Tewes and Volkmann [12, 13] and Meierling and Volkmann [7]. For more information concerning different generalizations of tournaments, the reader may be referred to the survey article of Bang-Jensen and Gutin [2].

The first result concerning longest paths in tournament-like digraphs is due to Rédei [8].

Theorem 1.2 (Rédei [8] 1934)**.** *Every tournament contains a Hamiltonian path.*

Note that Theorem 1.2 provides no information about the order of longest paths through given arcs of the tournament. For tournament-like digraphs this problem has been investigated by Volkmann [14, 15, 16] and Gutin and Yeo [5]. In 2002, Volkmann [16] proved that every arc of a strongly connected tournament of order n (even every arc of a strongly connected n-partite tournament) is contained in a path of order $\lceil (n+3)/2 \rceil$.

Theorem 1.3 (Volkmann [16] 2002)**.** *Every arc of a strong n-partite tournament belongs to a path of order* $\lceil \frac{n+3}{2} \rceil$.

The following example shows that this is no longer valid for strongly connected in-tournaments.

Example 1.4 (Volkmann [15] 2000)**.** *Let D be the in-tournament that consists of the cycle $x_1 x_2 \dots x_n x_1$ together with the arcs $x_1 x_i$ for $3 \leq i \leq n-1$. Then D is strong, has order n and the longest path through the arc $x_1 x_i$ is only of order $n - i + 2$.*

Let $m \in \{4, 5\}$ and let D be a strongly connected in-tournament of order $n \geq 2m - 2$ such that every arc belongs to a path of order at least m. In 2000, Volkmann [15] showed that if D contains an arc e such that the longest path through e consists of exactly m vertices, then e is the only arc of D with that property.

Theorem 1.5 (Volkmann [15] 2000)**.** *Let $m \in \{4, 5\}$ and let D be a strong in-tournament of order $n \geq 2m - 2$. If D has an m-path arc but no k-path arc for $3 \leq k \leq m - 1$, then D has exactly one m-path arc.*

In addition, Volkmann formulated the following conjecture.

Conjecture 1.6 (Volkmann [15] 2000)**.** *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n \geq 2m - 2$. If D has an m-path arc but no k-path arc for $3 \leq k \leq m - 1$, then D has exactly one m-path arc.*

The next example shows that the condition $n \geq 2m - 2$ in Conjecture 1.6 is best possible.

Example 1.7 (Volkmann [15] 2000)**.** *Let $m \geq 4$ be an integer and let D be the strong in-tournament that consists of the cycle $C = x_1 x_2 \dots x_{m-1} y_1 y_2 \dots y_{m-2} x_1$ such that $x_1 \to C[x_2, x_{m-1}]$ and $x_{m-1} \to C[y_1, y_{m-2}]$. Then D is of order $2m - 3$ and has the two m-path arcs $x_1 x_{m-1}$ and $x_{m-1} y_{m-2}$.*

We would also like to add that Conjecture 1.6 is not true if we consider the more general class of *in-semicomplete digraphs*, where there might be cycles of length two, instead of the class of in-tournaments.

Example 1.8. *Let D be the in-semicomplete digraph on 6 vertices defined by the 6-cycle $x_1x_2x_3x_4x_5x_6x_1$ and the additional arcs $x_1x_3, x_1x_4, x_4x_1, x_4x_6, x_5x_1$. Then $|V(D)| = 6 = 2 \cdot 4 - 2$ and every arc of D belongs to a path of order 4. But D contains two 4-path arcs, namely x_1x_4 and x_4x_1.*

In Section 4 we will validate Volkmann's conjecture. In addition, we will prove that if we ease the restrictions on the order of D to $n \geq 2m - 3$, the in-tournament D has at most two arcs as described in Conjecture 1.6. In doing so, we will also give a characterization of the in-tournaments with exactly two such arcs.

2 Preliminary results

The first result is a useful observation about the interaction of a cycle and a vertex which lies outside.

Theorem 2.1 (Bang-Jensen, Huang & Prisner [3] 1993)**.** *Let D be an in-tournament and let $C = u_1u_2 \ldots u_su_1$ be a cycle in D. If $|N^+(x, C)| \geq 1$ for a vertex $x \notin V(C)$, then either $x \to C$ or $u_i \to x \to u_{i+1}$ for an integer $1 \leq i \leq s$.*

The following results play an important role in our investigations.

Theorem 2.2 (Bang-Jensen, Huang & Prisner [3] 1993)**.** *An in-tournament has a Hamiltonian cycle if and only if it is strong.*

Theorem 2.3 (Bang-Jensen, Huang & Prisner [3] 1993)**.** *Let D be a strong in-tournament and let S be a minimal separating set of D.*

(a) If A and B are two distinct strong components of $D - S$, then either there is no arc between them or A weakly dominates B or B weakly dominates A. Furthermore, if A weakly dominates B, then $N^-(B, A)$ dominates B.

(b) If A and B are two distinct strong components of $D - S$ such that A weakly dominates B, then $N^-(b, A)$ induces a tournament for each $b \in B$.

(c) The strong components of $D - S$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i has an arc to D_{i+1} for $i = 1, 2, \ldots, p - 1$.

According to Theorem 2.3, we give the following definition.

Definition 2.4. *The unique labeling* $D_1, D_2, \ldots, D_p$ *of the strong components of* $D - S$ *as described in Theorem 2.3 is called* the strong decomposition of $D - S$. *We call* D_1 *the* initial *and* D_p *the* terminal component.

Theorem 2.5 (Bang-Jensen, Huang & Prisner [3] 1993)**.** *Let* D *be a strong in-tournament and let* S *be a minimal separating set of* D*. The strong decomposition of* $D - S$ *has the following properties.*

(a) If $x_i \rightarrow x_k$ *for* $x_i \in V(D_i)$ *and* $x_k \in V(D_k)$ *with* $1 \leq i \neq k \leq p$*, then* $x_i \rightarrow D_j$ *for every* $i + 1 \leq j \leq k$*.*

(b) The digraph $D - S$ *has a Hamiltonian path.*

(c) For every $s \in S$ *we have* $d^+(s, D_1) > 0$ *and* $d^-(s, D_p) > 0$*.*

The next observation is a first result on the order of a longest path through a given arc of an in-tournament.

Observation 2.6 (Volkmann [15] 2000)**.** *Let* uv *be an arbitrary arc of a strong in-tournament* D*. If* $D - u$ *or* $D - v$ *is strong, then* D *contains a Hamiltonian path starting with the arc* uv *or ending with the arc* uv*, respectively.*

Corollary 2.7. *If* uv *is an arc of a strong in-tournament* D *that does not belong to a Hamiltonian path, then both* u *and* v *are separating vertices of* D*.*

3 Structure of in-tournaments with an m-path arc

Let $m \geq 4$ be an integer. In this section we investigate strongly connected in-tournaments of order $n \geq 2m - 3$ that contain no k-path arc for $3 \leq k \leq m - 1$, but at least one m-path arc. Example 1.7 describes a class of in-tournaments with the above properties that contain two m-path arcs. Let D be a strongly connected in-tournament of order $n \geq 2m - 3$ that contains no k-path arc for $3 \leq k \leq m - 1$, but at least one m-path arc. In the following we will show that D has at most two m-path arcs. Taking a deeper look at the structure of D we shall show that if $n \geq 2m - 2$, the in-tournament D has only one m-path arc and we shall give a characterization of all strongly connected in-tournaments of order $n = 2m - 3$ that contain exactly two m-path arcs.

As an abbreviation we give the following notation which will be frequently used.

Notation 3.1. *Let D be a strong in-tournament with a separating vertex u and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$. Furthermore, for every index $i \in \{1,2,\ldots,p\}$ with $|V(D_i)| \geq 3$, let C_i be a Hamiltonian cycle of D_i, for every index $1 \leq i \leq p-1$, let $w_i \in V(D_i)$ be a vertex that dominates D_{i+1} and let $w_p \in V(D_p)$ be a negative neighbor of u. Then*

$$P = C_1[w_1^+, w_1]C_2[w_2^+, w_2] \ldots C_p[w_p^+, w_p]u$$

is a Hamiltonian path of D. For $1 \leq i \leq j \leq p$ let $P_{i,j}$ denote the subpath

$$P_{i,j} = C_i[w_i^+, w_i]C_{i+1}[w_{i+1}^+, w_{i+1}] \ldots C_j[w_j^+, w_j]$$

of P with initial vertex w_i^+ and terminal vertex w_j.

We begin with a first structural result.

Lemma 3.2. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order n such that D has no k-path arc for $3 \leq k \leq m-1$. Let uv be an m-path arc of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_p)$, then $n \leq 2m-3$.*

Proof. Let uv be an m-path arc of D such that $v \in V(D_p)$. Suppose that $n \geq 2m-2$. According to Corollary 2.5, the vertex u has a negative neighbor w_p in D_p which implies that $|V(D_p)| \geq 3$. Let $vx_1x_2\ldots x_tv$ be a Hamiltonian cycle of D_p, where $t \geq 2$. If $t \geq m-1$, the path $uvx_1x_2\ldots x_t$ is an $(m+1)$-path through uv, a contradiction. So $t \leq m-2$. It follows that

$$|V(D) - (V(D_p) \cup \{u\})| = n - (t+1) - 1 \geq n - m \geq m-2$$

and thus, $P_{1,p-1}w_puv$ is a path through uv of order at least $m+1$, a contradiction. $\square$

It remains to consider the case that v belongs to the terminal strong component of $D-u$ and D has order $n = 2m-3$.

Lemma 3.3. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n = 2m-3$ such that D has no k-path arc for $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_p)$ and $w = w_p$ is an in-neighbor of u in D_p, then uv is the unique m-path arc of D and D has the following structural properties.*

(a) $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| = m-3$;

(b) $|V(D_p)| = m-1$;

(c) $(D_p - w_p) \rightsquigarrow v \rightarrow w_p \rightsquigarrow (D_p - v)$*;*

(d) $u \rightarrow V(D) - \{u, w_p\}$*.*

Proof. Let uv be an m-path arc of D such that $v \in V(D_p)$.

(a) Since $uC_p[v, v^-]$ is a path through uv and uv is an m-path arc of D, it follows that $|V(D_p)| \leq m - 1$. Hence $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| \geq m - 3$. But if $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| > m - 3$, the path $P_{1,p-1}w_p uv$ is a path through uv of order at least $m + 1$, a contradiction. Hence, D has property (a).

(b) Property (a) immediately implies (b).

(c) Since D_p is strong and $|V(D_p)| \geq 3$, it follows that v has at least one positive neighbor in D_p and w_p has at least one negative neighbor in D_p. If v dominates a vertex $x \neq w_p$ in D_p, the path $P_{1,p-1}w_p uvx$ is a path of order $m+1$ through uv, a contradiction. So $v \rightarrow w_p$ and $(D_p - w_p) \rightsquigarrow v$. If w_p is dominated by a vertex $x \neq v$ in D_p, the path $P_{1,p-1}xw_p uv$ is an $(m+1)$-path through uv, a contradiction. So $w_p \rightsquigarrow (D_p - v)$ and the proof of (c) is complete.

(d) If $|N^-(u, D_p)| \geq 2$, let $x \notin \{v, w_p\}$ be a negative neighbor of u in D_p. Then $P_{1,p-1}xuvw_p$ is an $(m+1)$-path through uv, a contradiction. So u has exactly one negative neighbor in D_p, the vertex w_p. Note that property (c) implies particularly that w_p is the successor of v on C_p. Using Theorem 2.1, it follows that $u \rightarrow (D_p - w_p)$. Note that

$$P = P_{1,p} = C_1[w_1^+, w_1]C_2[w_2^+, w_2] \ldots C_p[w_p^+, w_p]$$

is a Hamiltonian path of $D - u$ such that $u \rightarrow P[w_p^+, w_p^-]$. Using the in-tournament property of D, either $u \rightarrow D_1, D_2, \ldots, D_{p-1}$ or there exists a vertex x on P such that $x \rightarrow u \rightarrow x_P^+$. But in the latter case $xuC_p[v, v^-]$ is an $(m+1)$-path through uv, a contradiction. This means that D fulfills (d).

It remains to show that uv is the unique m-path arc of D. Every arc xy, where $x \in V(D_i)$ and $y \in V(D_j)$ for $i + 1 \leq j \leq p - 1$, belongs to the path

$$xC_j[y, w_j]P_{j+1,p}$$

of order at least

$$|\{x, y\}| + \sum_{\ell=j+1}^{p} |V(D_\ell)| \geq 2 + |V(D_p)| = m + 1.$$

If xy is an arc from D_i to D_p for an integer $i \leq p-1$, the path

$$uxC_p[y, y^-]$$

through uv has order

$$|\{u, x\}| + |V(D_p)| = m + 1.$$

In the case that xy is an arc of D such that $x, y \in V(D_i)$, where $i \leq p-1$, we see that

$$C_p[w_p^+, w_p]uxy$$

is a path of order

$$|V(D_p)| + |\{u, x, y\}| = m + 2$$

through xy. Every arc xy in D_p belongs to the path

$$uP_{1,p-1}xyy^+$$

of order

$$|\{u, x, y, y^+\}| + \sum_{\ell=1}^{p-1} |V(D_\ell)| = m + 1.$$

If uy is an arc from u to D_i for $1 \leq i \leq p-1$, the path

$$uC_i[y, w_i]P_{i+1,p}$$

through uy has order at least

$$|\{u, y\}| + \sum_{\ell=i+1}^{p} |V(D_\ell)| \geq 2 + |V(D_p)| = m + 1.$$

In the case that uy is an arc from u to D_p such that $y \neq v$, the path

$$P_{1,p-1}w_puyy^+$$

is a path of order

$$|\{w_p, u, y, y^+\}| + \sum_{\ell=1}^{p-1} |V(D_\ell)| = m + 1$$

through uy. Finally the arc w_pu belongs to the Hamiltonian path $P_{1,p}u$ of D. Since we have discussed all possible arcs, the proof is complete. $\square$

The next result shows that the class of in-tournaments treated in this paper can be divided in two subclasses.

Lemma 3.4. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n = 2m - 3 + c$, where $c \geq 0$, that contains no k-path arc for $3 \leq k \leq m-1$, but an m-path arc uv. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $v \in V(D_s)$ for $1 \leq s \leq p-1$, then $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$ and, in addition, D has one of the following properties.*

I. $|V(D_s) \cup V(D_{s+1}) \cup \ldots \cup V(D_p)| = m - 1$ *and*

- *(a)* $u \to D_1, D_2, \ldots, D_s$;
- *(b)* $N^-(D_i, D_j) = \emptyset$ *for every* $s+1 \leq i \leq p$ *and* $1 \leq j \leq s-1$;
- *(c)* $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{s-1})| \geq m - 3 + c$.

II. $|V(D_s) \cup V(D_{s+1}) \cup \ldots \cup V(D_p)| = m - k$, *where* $2 \leq k \leq m-2$, *and there exists an index* $1 \leq r \leq s-1$ *and a vertex* $w_r \in V(D_r)$ *such that*

- *(a)* $u \to D_{r+1}, D_{r+2}, \ldots, D_s$;
- *(b)* $w_r \to D_{s+1}$;
- *(c)* $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_r)| = k - 1$;
- *(d)* $|V(D_{r+1}) \cup V(D_{r+2}) \cup \ldots \cup V(D_{s-1})| \geq m - 3 + c$.

Proof. The path

$$P_{s+1,p}uC_s[v, v^-]$$

contains the arc uv and has order $\sum_{j=s}^{p} |V(D_j)| + 1$. Since uv is an m-path arc, it follows that $\sum_{j=s}^{p} |V(D_j)| \leq m - 1$. So we consider the following two cases.

Case I. Suppose that $\sum_{j=s}^{p} |V(D_j)| = m - 1$. This immediately implies (c).

Assume that there exists a vertex $v \neq z \in V(D_s)$ that dominates u or w_j, where $s+1 \leq j \leq p$. Then $z \to D_{s+1}$ and hence

$$P_{1,s-1}C_s[v^+, z]P_{s+1,p}uv$$

is a path through uv of order at least

$$\sum_{\ell=1}^{s-1} |V(D_\ell)| + |V(D_p)| + |\{z, u, v\}| \geq (m - 3 + c) + 4 \geq m + 1,$$

a contradiction. So $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$ and $u \to D_s$.

Assume that there exists a vertex $z \in V(D_i)$ that dominates u or w_j, where $1 \leq i \leq s-1$ and $s+1 \leq j \leq p$. It follows that $z \to D_{s+1}$ and thus,

$$zP_{s+1,p}uC_s[v, v^-]$$

is a path of order

$$\sum_{\ell=s}^{p} |V(D_\ell)| + |\{z, u\}| = (m-1) + 2 = m+1$$

through uv, a contradiction. Hence D fulfills (a) and (b) which completes the proof of Case I.

Case II. Suppose that $\sum_{j=s}^{p} |V(D_j)| = m-k$, where $2 \leq k \leq m-2$. If D has no arc leading from $\bigcup_{i=1}^{s} V(D_i) \setminus \{v\}$ to $\{u\} \cup \bigcup_{j=s+1}^{p} V(D_j)$, it is easy to check that a longest path through uv has order $m-1$, a contradiction. So there exists a vertex $v \neq z \in V(D_r)$, where $r \leq s$, that dominates w_{s+1}. If $r = s$, the path

$$P_{1,s-1} z P_{s+1,p} uv$$

through uv contains at least

$$(2m-3+c) - (m-k) + 2 = m+k+c-1 \geq m+1$$

vertices, a contradiction. It follows that $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$, that $u \to D_s$ and that $r < s$. Under the condition that $N^-(D_{s+1}, D_r) \neq \emptyset$ we choose the maximal index r. This choice immediately implies that $u \to D_{r+1}, D_{r+2}, \ldots, D_{s-1}$ and that, without loss of generality, $w_r \to D_{s+1}$. So (a) and (b) are true. But then

$$P_{1,r} P_{s+1,p} u C_s[v, v^-]$$

is a path through uv of order $\sum_{j=1}^{r} |V(D_j)| + (m-k) + 1$. Since uv is an m-path arc in D, this implies that $\sum_{j=1}^{r} |V(D_j)| \leq k-1$. Note that actually

$$\sum_{j=1}^{r} |V(D_j)| = k-1,$$

since otherwise a longest path through uv contains less than m vertices, a contradiction. So (c) is proved. But (c) immediately implies that

$$\sum_{j=r+1}^{s-1} |V(D_j)| \geq m-3+c$$

which shows (d) and completes the proof of Case II. □

4 Applications of the structural results

Differentiating by the properties I and II elaborated in Section 3, we shall now show that if a strongly connected in-tournament D of order at least $2m-3$ contains an m-path arc uv, but no k-path arc for $3 \leq k \leq m-1$, the digraph D has at most two m-path arcs. We start with in-tournaments that belong to class I.

Lemma 4.1. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n = 2m-3+c$, where $c \geq 0$, such that D has no k-path arc for $3 \leq k \leq m-1$. Let uv be an m-path arc of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_s)$, where $1 \leq s \leq p-1$, and D has property* I *as described in Lemma 3.4, then D contains at most two m-path arcs.*

Proof. We shall show that every arc $xy \neq uv$ of D is contained in a path of order at least $m+1$ with two exceptions (cf. Case 2 and Case 5) which cannot occur simultaneously.

Case 1. Suppose that $x \in V(D_i)$ and $y \in V(D_{i+1})$. If $i < s$, the arc xy belongs to the Hamiltonian path

$$P_{1,i-1}C_i[x^+,x]C_{i+1}[y,w_{i+1}]P_{i+2,p}uC_{i+1}[w_{i+1}^+,y^-].$$

If $i \geq s$, the arc xy belongs to the path

$$P_{1,i-1}C_i[x^+,x]C_{i+1}[y,w_{i+1}]P_{i+2,p}u.$$

Assume that this path has order less or equal m. In this case we conclude that $\sum_{j=1}^{i-1}|V(D_j)| = m-3$, $i = s$, $|V(D_i)| = |V(D_{i+1})| = 1$ and $p = s+1$. But the latter two observations imply that

$$m-1 = \sum_{j=s}^{p}|V(D_j)| = 2 \Leftrightarrow m = 3,$$

a contradiction.

Case 2. Suppose that $x \in V(D_i)$ and $y \in V(D_j)$, where $i+2 \leq j$. If $i < s$, the arc xy belongs to the Hamiltonian path

$$P_{1,i-1}C_i[x^+,x]C_j[y,w_j]P_{j+1,p}uP_{i+1,j-1}C_j[w_j^+,y^-].$$

If $i \geq s$, the arc xy belongs to the path

$$P_{1,i-1}C_i[x^+,x]C_j[y,w_j]P_{j+1,p}u.$$

Assume that this path has order less or equal m. In this case we conclude that $\sum_{j=1}^{i-1}|V(D_j)| = m-3$. So $c = 0$ which implies that $n = 2m-3$. Furthermore, we observe that $V(D_s) = V(D_i) = \{x\}$ and $V(D_p) = V(D_j) = \{y\}$ which means that $x = v$ and $y = w_p$. If D has an arc uz such that $z \in V(D_i)$ for an index $s+1 \le i \le p-1$, the path $P_{1,s-1}xyuz$ is a path through xy and if D has an arc zu such that $z \in V(D_i)$ for an index $s+1 \le i \le p-1$, the path $zuP_{1,s-1}xy$ is a path through xy. Both paths have order

$$\sum_{\ell=1}^{s-1} |V(D_\ell)| + |\{x, y, u, z\}| \ge (m-3) + 4 \ge m+1.$$

So assume that no such arc exists in D. Then $xy = vw_p$ is an m-path arc in D.

Case 3. Suppose that $x, y \in V(D_i)$. If $i \ge s$, the arc xy belongs to

$$uP_{1,i-1}C_i[x^-, x]y,$$

a path of order at least

$$\sum_{\ell=1}^{s-1} |V(D_\ell)| + |\{x^-, x, u, y\}| \ge (m-3+c) + 4 \ge m+1.$$

If $i < s$, the arc xy belongs to $P_{s,p}uxy$, a path of order

$$\sum_{\ell=s}^{p} |V(D_\ell)| + |\{u, x, y\}| = m+2.$$

Case 4. Suppose that $x = u$ and $y \in V(D_j)$. Due to Lemmas 2.3 and 3.2 we may assume that $j \le p-1$. If $j \ge s$ and $y \ne w_j$, the arc xy belongs to

$$P_{1,j-1}C_j[y^+, w_j]P_{j+1,p}uy,$$

a path of order at least

$$\sum_{\ell=1}^{j-1} |V(D_\ell)| + |V(D_p)| + |\{w_j, u, y\}| \ge (m-3+c) + 4 \ge m+1.$$

If $j < s$, the arc xy belongs to $P_{j+1,p}uy$, a path of order at least

$$\sum_{\ell=j+1}^{p} |V(D_\ell)| + |\{u, y\}| \ge (m-1) + 2 = m+1.$$

It remains to check the case that $j > s$ and $y = w_j$ is the only vertex in D_j that dominates D_{j+1}. If D has no arc leading from $\bigcup_{i=1}^{j-1} V(D_i)$ to $\{u\} \cup \bigcup_{i=j+1}^{p} V(D_i)$, a longest path through uy has order at most $m-2$, a contradiction. So there exists a vertex $w \in V(D_i)$, where $i \leq j-1$, that dominates D_{j+1}. If $i < s$, note that $w \to D_{s+1}$ and thus,

$$wP_{s+1,p}uC_s[v, v^-]$$

is a path of order

$$\sum_{\ell=s}^{p} |V(D_\ell)| + |\{u, w\}| = (m-1) + 2 = m+1$$

through uv, a contradiction. If $i \geq s$, the arc uy belongs to

$$P_{1,i-1}C_i[w^+, w]P_{j+1,p}uC_j[y, y^-],$$

a path of order at least

$$\sum_{\ell=1}^{i-1} |V(D_\ell)| + |V(D_i)| + |V(D_p)| + |\{u, y\}| \geq (m-3+c) + 4 \geq m+1.$$

Case 5. Suppose that $x \in V(D_i)$ and $y = u$. Since $u \to D_1, D_2, \ldots, D_s$, we conclude that $i \geq s+1$. But then the arc xu belongs to

$$P_{1,i-1}C_i[x^+, x]u,$$

a path of order at least

$$\sum_{\ell=1}^{i-2} |V(D_\ell)| + |V(D_{i-1})| + |V(D_i)| + |\{u\}| \geq (m-3+c) + 3 \geq m.$$

Assume that this path has order less or equal m. In this case we conclude that $\sum_{\ell=1}^{i-1} |V(D_\ell)| = m-3$. So $c = 0$ which implies that $n = 2m-3$. Furthermore, we observe that $V(D_s) = V(D_{i-1}) = \{v\}$ and $V(D_{s+1}) = V(D_i) = \{x\}$. If D has an arc uz such that $z \in V(D_i)$ for an index $i \geq s+2$, the path $P_{1,s-1}vxuz$ is a path of order

$$\sum_{\ell=1}^{s-1} |V(D_\ell)| + |\{v, x, u, z\}| \geq (m-3) + 4 \geq m+1$$

through xu. So assume that no such arc exists in D. Then $xy = w_{s+1}u$ is an m-path arc in D.

It remains to show that the two exceptions in Case 2 and Case 5 cannot occur simultaneously. So assume that D is an in-tournament with $|V(D_s)| = |V(D_{s+1})| = |V(D_p)| = 1$ and the three m-path arcs uw_s, w_sw_p and $w_{s+1}u$. Then $w_{s+1}uP_{1,s-1}w_sw_p$ is an $(m+1)$-path through $w_{s+1}u$, a contradiction. Since we have discussed all possible arcs, the proof is complete. □

We now turn our attention to in-tournaments belonging to class II.

Lemma 4.2. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n = 2m - 3 + c$, where $c \geq 0$, such that D has no k-path arc for $3 \leq k \leq m-1$. Let uv be an m-path arc of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $v \in V(D_s)$, where $1 \leq s \leq p-1$, and D has property* II *as described in Lemma 3.4, then uv is the unique m-path arc in D.*

Proof. We shall show that every arc $xy \neq uv$ belongs to a path of order at least $m + 1$.

Case 1. Suppose that xy is an arc of D such that $x \in V(D_i)$ and $y \in V(D_{i+1})$. If $i < r$ or if $i \geq s$, the arc xy belongs to

$$P_{1,i-1}C_i[x^+, x]C_{i+1}[y, w_{i+1}]P_{i+2,p}u,$$

a path of order at least

$$\begin{aligned}\sum_{\ell=i+2}^{p} |V(D_\ell)| + |\{x, y, u\}| &\geq \sum_{\ell=r+1}^{s-1} |V(D_\ell)| + |V(D_p)| + |\{x, y, u\}| \\ &\geq (m - 3 + c) + 4 \geq m + 1\end{aligned}$$

and

$$\sum_{\ell=1}^{i-1} |V(D_\ell)| + |V(D_i)| + |\{u, y\}| \geq (m - 3 + c) + (k - 1) + 3 \geq m + 1,$$

respectively. If $r \leq i \leq s - 1$, we can extend the path above by

$$C_{i+1}[w_{i+1}^+, y^-]$$

to a Hamiltonian path through xy.

Case 2. Suppose that $x \in V(D_i)$ and $y \in V(D_j)$ for $j \geq i + 2$. If $i \geq s$, the arc xy belongs to

$$P_{1,i-1}C_i[x^+, x]C_j[y, w_j]P_{j+1,p}u,$$

a path of order at least

$$\sum_{\ell=1}^{i-1} |V(D_\ell)| + |V(D_i)| + |\{u, y\}| \geq (m - 3 + c) + (k - 1) + 3 \geq m + 1.$$

If $i \leq r$ and $j \leq r$, the arc xy belongs to

$$xC_j[y, w_j]P_{j+1,p}u,$$

a path of order at least

$$\sum_{\ell=j+1}^{p} |V(D_\ell)| + |\{u,x,y\}| \geq (m-3+c)+(m-k)+3 \geq 2m-k \geq m+2.$$

If $i \leq r$ and $j > r$, the arc xy belongs to

$$xC_j[y,w_j]P_{j+1,p}uP_{r+1,j-1}C_j[w_j^+,y^-],$$

a path of order at least

$$\sum_{\ell=r+1}^{p} |V(D_\ell)| + |\{x,u\}| \geq (m-3+c)+(m-k)+2 \geq 2m-k-1 \geq m+1.$$

Finally, if $r+1 \leq i \leq s-1$, the path

$$P_{1,i-1}C_i[x^+,x]C_j[y,w_j]P_{j+1,p}uP_{i+1,j-1}C_j[w_j^+,y^-]$$

is a Hamiltonian path through xy.

Case 3. Suppose that $x=u$ and $y \in V(D_j)$. If $j \leq r$, the arc uy belongs to

$$uC_j[y,w_j]P_{j+1,p},$$

a path of order at least

$$\sum_{\ell=j+1}^{p} |V(D_\ell)| + |\{u,y\}| \geq (m-3+c)+(m-k)+2 \geq 2m-k-1 \geq m+1.$$

If $r < j < s$, the arc uy belongs to

$$P_{1,r}P_{s+1,p}uC_j[y,w_j]P_{j+1,s},$$

a path of order at least

$$\sum_{\ell=1}^{r} |V(D_\ell)| + \sum_{\ell=j+1}^{p} |V(D_\ell)| + |\{u,y\}| \geq (k-1)+(m-k)+2 = m+1.$$

If $j=s$, it follows that $y \neq v$ and thus, the arc xy belongs to

$$P_{1,s-1}C_s[y^+,w_s]P_{s+1,p}uy,$$

a path of order at least

$$\sum_{\ell=1}^{s-1} |V(D_\ell)| + |\{w_s,u,y\}| \geq (m-3+c)+(k-1)+3 \geq m+k-1 \geq m+1.$$

Finally, if $j > s$, either $y \neq w_j$ or $y = w_j$ and w_j is the only vertex in D_j that dominates D_{j+1}. In the first case

$$P_{1,j-1}C_j[y^+, w_j]P_{j+1,p}uy$$

is a path through uy of order at least

$$\begin{aligned}\sum_{\ell=1}^{j-2}|V(D_\ell)| + |V(D_{j-1})| + |\{w_j, u, y\}| &\geq (m-3+c) + (k-1) + 4\\ &\geq m + k \geq m + 2.\end{aligned}$$

In the second case we observe the following. If D does not have an arc leading from $\bigcup_{i=r+1}^{j-1} V(D_i)$ to $\{u\} \cup \bigcup_{i=j+1}^{p} V(D_i)$, a longest path through uy has order at most $m-1$, a contradiction. So there exists a vertex $z \in V(D_i)$, where $r+1 \leq i \leq j-1$, that dominates D_{j+1}.

If $i < s$, then $z \to D_{s+1}$ and thus,

$$P_{1,i-1}C_i[z^+, z]P_{s+1,p}uC_s[v, v^-]$$

is a path through uv of order at least

$$\sum_{\ell=1}^{i-1}|V(D_\ell)| + |V(D_i)| + \sum_{\ell=s}^{p}|V(D_\ell)| + |\{u\}| \geq (k-1) + (m-k) + 2 = m+1,$$

a contradiction.

If $i \geq s$, then $z \to D_{j+1}$ and thus,

$$P_{1,i-1}C_i[z^+, z]P_{j+1,p}uC_j[y, y^-]$$

is a path through uy of order at least

$$\sum_{\ell=1}^{i-1}|V(D_\ell)| + |V(D_i)| + \sum_{\ell=j}^{p}|V(D_\ell)| + |\{u\}| \geq (k-1) + (m-3+c) + 3 \geq m+1,$$

again a contradiction.

Case 4. Suppose that $x, y \in V(D_i)$. If $i \geq s+1$, the arc xy belongs to $P_{1,i-1}xy$, a path of order at least

$$\begin{aligned}\sum_{\ell=1}^{i-2}|V(D_\ell)| + |V(D_{i-1})| + |\{x, y\}| &\geq (m-3+c) + (k-1) + 3\\ &\geq m + k - 1 \geq m + 1.\end{aligned}$$

If $r+1 \le i < s$, the arc xy belongs to

$$P_{1,r}P_{s,p}uP_{r+1,i-1}xy,$$

a path of order at least

$$\sum_{\ell=1}^{r}|V(D_\ell)| + \sum_{\ell=s}^{p}|V(D_\ell)| + |\{u,x,y\}| \ge (k-1)+(m-k)+3 = m+2.$$

If $i = s$, the arc xy belongs to

$$P_{1,r}P_{s+1,p}uP_{r+1,s-1}xy,$$

a path of order

$$2m-3+c-|V(D_s)|+2 \ge 2m-1-(m-2) = m+1.$$

Finally, if $1 \le i \le r$, we observe the following. Firstly, any path from y to any vertex $z \in V(D_i)$ that dominates D_{i+1} uses the vertex x and secondly, any path from u to x uses the vertex y, since otherwise it is easy to check that xy belongs to a path of order at least $m+1$. It follows that every vertex of a longest path through xy belongs to $V(D_1) \cup V(D_2) \cup \ldots \cup V(D_i)$. But this implies that the longest path through xy is of order at most $k-1 \le m-1$, a contradiction.

Case 5. Suppose that $y = u$ and $x \in V(D_i)$. It follows that either $i \le r$ or $i > s$. In the first case the arc xy belongs to

$$P_{1,i-1}C_i[x^+,x]uP_{r+1,p},$$

a path of order at least

$$\sum_{\ell=r+1}^{p}|V(D_\ell)| + |\{x,u\}| \ge (m-3+c)+(m-k)+2 \ge m+1$$

and in the second case xy belongs to $P_{1,i-1}xu$, a path of order at least

$$\sum_{\ell=1}^{i-2}|V(D_\ell)| + |V(D_{i-1})| + |\{x,u\}| \ge (m-3+c)+(k-1)+3 \ge m+1.$$

Since we have discussed all possible arcs, the proof is complete. □

Combining the lemmas above, we get the following characterization of strongly connected in-tournaments that contain an m-path arc.

Theorem 4.3. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n \geq 2m - 3$ such that D has no k-path arc for $3 \leq k \leq m - 1$.*

(a) If $n \geq 2m - 2$, then D has at most one m-path arc.

(b) If $n = 2m - 3$, then D has at most two m-path arcs. In addition, D has exactly two m-path arcs uv and xy if and only if D has property I *as described in Lemma 3.4 and $V(D_s) = \{v\}$ and either*

  - *(i) $v = x$, $V(D_p) = \{y\}$ and there exists no arc between u and D_j for every index $s + 1 \leq j \leq p - 1$ or*
  - *(ii) $y = u$, $V(D_{s+1}) = \{x\}$ and there exists no arc from D_i or u to D_j for every index $1 \leq i \leq s$ and $s + 2 \leq j \leq p$.*

(Here $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, where $p \geq 2$.)

As a corollary of Theorem 4.3.(a) Volkmann's Conjecture 1.6 is solved in positive.

Corollary 4.4. *Let $m \geq 4$ be an integer and let D be a strong in-tournament of order $n \geq 2m - 2$. If D has an m-path arc but no k-path arc for $3 \leq k \leq m - 1$, then D has exactly one m-path arc.*

Acknowledgments

I would like to thank L. Volkmann for pointing me to his Conjecture 1.6 in [15]. Furthermore, I would like to thank the anonymous referees for their suggestions that helped to improve the presentation of the paper.

References

[1] J. Bang-Jensen, *Locally semicomplete digraphs: a generalization of tournaments*, J. Graph Theory **14** (1990), no. 3, 371–390.

[2] J. Bang-Jensen and G. Gutin, *Generalizations of tournaments: a survey*, J. Graph Theory **28** (1998), no. 4, 171–202.

[3] J. Bang-Jensen, J. Huang, and E. Prisner, *In-tournament digraphs*, J. Combin. Theory Ser. B **59** (1993), no. 2, 267–287.

[4] Y. Guo, *Locally Semicomplete Digraphs*, Ph.D. thesis, RWTH Aachen University, 1995, 92 p.

[5] G. Gutin and A. Yeo, *Solution of a conjecture of Volkmann on the number of vertices in longest paths and cycles of strong semicomplete multipartite digraphs*, Graphs Combin. **17** (2001), no. 3, 473–477.

[6] J. Huang, *Tournament-Like Oriented Graphs*, Ph.D. thesis, Simon Fraser University, 1992.

[7] D. Meierling and L. Volkmann, *All 2-connected in-tournaments that are cycle complementary*, Discrete Math. **308** (2008), 2115–2133, doi:10.1016/j.disc.2006.12.008.

[8] L. Rédei, *Ein kombinatorischer Satz*, Acta Litt. Sci. Szeged **7** (1934), 39–43.

[9] M. Tewes, *In-Tournaments and Semicomplete Multipartite Digraphs*, Ph.D. thesis, RWTH Aachen University, 1999, 114 p.

[10] M. Tewes, *Pancyclic in-tournaments*, Discrete Math. **233** (2001), no. 1-3, 193–204, Graph theory (Prague, 1998).

[11] M. Tewes, *Pancyclic orderings of in-tournaments*, Discrete Appl. Math. **120** (2002), no. 1-3, 239–249, Sixth Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1999).

[12] M. Tewes and L. Volkmann, *On the cycle structure of in-tournaments*, Australas. J. Combin. **18** (1998), 293–301.

[13] M. Tewes and L. Volkmann, *Vertex pancyclic in-tournaments*, J. Graph Theory **36** (2001), no. 2, 84–104.

[14] L. Volkmann, *Longest paths in semicomplete multipartite digraphs*, Discrete Math. **199** (1999), no. 1-3, 279–284.

[15] L. Volkmann, *Longest paths through an arc in strongly connected in-tournaments*, Australas. J. Combin. **21** (2000), 95–106.

[16] L. Volkmann, *Longest paths through an arc in strong semicomplete multipartite digraphs*, Discrete Math. **258** (2002), no. 1-3, 331–337.