# Local Tournaments and In-Tournaments

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften
der Rheinisch-Westfälischen Technischen Hochschule Aachen
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
genehmigte Dissertation

vorgelegt von

Diplom-Mathematiker

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Tag der mündlichen Prüfung: 28.11.2007

# Danksagung

Bedanken möchte ich mich an dieser Stelle bei allen, die mich bei der Erstellung dieser Arbeit unterstützt haben. Mein besonderer Dank gilt meinem Betreuer Herrn Professor Dr. Lutz Volkmann, in dessen Vorlesungen und Seminaren mein Interesse für die Graphentheorie geweckt wurde. Zahlreiche Gespräche und fruchtbare Diskussionen schufen ein besonderes Arbeitsklima und waren Grundlage für viele wertvolle Hinweise und Anregungen.

Des Weiteren danke ich dem Inhaber des Lehrstuhls II für Mathematik, Herrn Professor Dr. Eberhard Triesch, der sich als Korreferent dieser Dissertation zur Verfügung stellte.

Weiterer Dank gilt allen Mitarbeitern des Lehrstuhls II für Mathematik, die zur freundschaftlichen Arbeitsatmosphäre beitrugen. In diesem Zusammenhang danke ich auch unserer Sekretärin Frau Hannelore Volkmann, die einen großen Anteil daran hatte.

Für das Korrekturlesen und ihre Geduld mit meinen Problemen aus der Welt der Graphentheorie möchte ich besonders Frau Anja Schutzbach danken.

Schließlich danke ich meinen Eltern für ihre Unterstützung.

Aachen, im Dezember 2007 Dirk Meierling

# Preface

Tournaments constitute perhaps the most well-studied class of directed graphs. One of the reasons for the interest in the theory of tournaments is the monograph *Topics on Tournaments* [58] by Moon published in 1968, covering all results on tournaments known up to this time. In particular, three results deserve special mention: in 1934 Rédei [60] proved that every tournament has a directed Hamiltonian path, in 1959 Camion [27] showed that every strongly connected tournament has a directed Hamiltonian cycle and in 1966 Moon [57] published his famous theorem which says that every strongly connected tournament on $n$ vertices is vertex pancyclic, i.e. every vertex is contained in a directed cycle of length $3, 4, \ldots, n$. The latter result is a strong improvement of Camion's theorem and can be considered a milestone in the investigation of the cycle structure of tournaments. In the following years survey articles dealing with tournaments were contributed by Beineke and Wilson [20] in 1975, Reid and Beineke [63] in 1978, Beineke [19] and Bermond and Thomassen [21] in 1981, Bang-Jensen and Gutin [10] and Reid [62] in 1996. All these articles kept the interest of mathematicians working in this field growing.

There are various generalizations of tournaments: local tournaments and locally semicomplete digraphs, in-tournaments and locally in-semicomplete digraphs, multipartite tournaments and semicomplete multipartite digraphs, quasi-transitive digraphs and several others. For a comprehensive overview of multipartite tournaments and semicomplete multipartite digraphs the reader may be referred to the survey articles of Gutin [40] and Volkmann [75, 78], and for a summary of the remaining classes to the article of Bang-Jensen and Gutin [11]. It is a natural question to ask which of the results for tournaments can be extended to one of its generalized classes. In view of Rédei's and Moon's theorems, the path and cycle structure are specifically of interest. In this thesis, two different generalizations of tournaments are considered, namely the class of local tournaments and the class of in-tournaments. For the sake of simplicity, directed paths and directed cycles are called paths and cycles throughout this thesis.

A digraph without loops, multiple arcs and cycles of length two is called a *local tournament* if the set of in-neighbors as well as the set of out-neighbors of every vertex induces a tournament. This class, or more generally the class of *locally semicomplete digraphs* whose members might have cycles of length two, was introduced by Bang-Jensen [3] in 1990. Three years later, Bang-Jensen, Huang and Prisner [16] introduced a further generalization of local tournaments, the class of *in-tournaments*, in claiming adjacency only for vertices that have a common out-neighbor. Hence local tournaments and in-tournaments are generalizations of tournaments in such a way that the general adjacency of vertices is transferred to only those pairs of vertices that share a local property. Other such classes are, for example, quasi-transitive digraphs (cf. [14]) and arc-locally semicomplete digraphs (cf. [7]).

Ever since Bang-Jensen introduced the class of local tournaments, a lot of research has been done in this field. In particular the Ph.D. theses of Huang [45] and Guo [33] handled this subject in detail.

Concerning in-tournaments and, in particular, the path and cycle structure of in-tournaments, some initial fundamental results were presented by Bang-Jensen, Huang and Prisner [16]. Further work on pancyclicity and the cycle structure of in-tournaments has been done by Tewes and Volkmann [68, 69], Tewes [66, 67] and Peters and Volkmann [59]. The reader may be referred to the Ph.D. thesis of Tewes [65] for an overview of this subject.

In this thesis we study the path and cycle structure of local tournaments and in-tournaments and the related topic of vertex deletion. The required terminology and notation as well as the basic structural properties of local tournaments and in-tournaments are established in Chapter 1. The remaining part of this thesis is subdivided in three parts.

In the first part consisting of Chapters 2 to 4 we investigate the path structure of local tournaments and in-tournaments. The first problem we deal with in Chapter 2 is to develop the structure necessary for a local tournament to be not arc-traceable, i.e. to contain an arc that does not belong to a Hamiltonian path. Using this structure we give various sufficient criteria for a local tournament to be arc-traceable and we present examples that show the sharpness of our conditions. Our results extend those of Busch [25] and Busch, Jacobson and Reid [26] for tournaments. In the next two chapters — Chapters 3 and 4 — we focus on shorter paths in local tournaments and in-tournaments. We show that a strongly connected in-tournament on at least $2m-2$ vertices, where $m \geq 4$, with the property that each arc lies on a path of order at least $m$, contains at most one $m$-path arc, i.e. an arc such that the longest path through it is of order $m$. This confirms a conjecture of Volkmann [74]. Subsequently we study two variations of the above problem.

Firstly we lower the requirement on the order of the in-tournament in question to $n \geq 2m - 3$ (Chapter 3) and secondly we consider local tournaments instead of in-tournaments, allowing the local tournament to be of order as low as $n \geq \frac{3m-2}{2}$ though (Chapter 4). Finally we present examples that show the sharpness of our results in terms of order $n$.

The second part of this thesis consists of two chapters in which we investigate the following problem: How many non-separating vertices does a strongly connected local tournament or in-tournament have in terms of minimum vertex degree? In Chapter 5 we prove that every strongly connected in-tournament $D$ on $n$ vertices with minimum degree $\delta(D) \geq p \geq 2$ has at least $k = \min\{n, 4p-2\}$ non-separating vertices. If $n \geq 4p$ or if $n \geq 4p + 1$ and $p \geq 3$, this proposition can be improved to $k = 4p - 1$ and $k = 4p$, respectively, unless $D$ is a member of a well-described exceptional family. The results in this chapter generalize those of Kotani [50] for tournaments and of Meierling and Volkmann [55] for local tournaments. The topic of Chapter 6 is the problem of vertex deletion, where we ask for the existence of two distinct vertices in a strongly connected local tournament whose removal preserves the strong connectivity of the digraph. We characterize all local tournaments with exactly two such vertices thereby generalizing a result of Las Vergnas [51] for tournaments.

The third part of this thesis is devoted to the cycle structure of local tournaments and in-tournaments. A reformulation of the results of Chapter 6 shows that we characterized all local tournaments on $n$ vertices with exactly two cycles of length $n - 1$. Using the parameter $g(D)$, called quasi-girth of a local tournament $D$, Bang-Jensen, Guo, Gutin and Volkmann [8] proved that every strongly connected local tournament is vertex $(g(D) + 1)$-pancyclic. In Chapter 7 we investigate how many cycles of length $r$, where $g(D) + 1 \leq r \leq n - 1$, a strongly connected local tournament has at the least. This problem has already been studied and solved completely by Moon [57] and Las Vergnas [51] for tournaments and our results generalize those of Las Vergnas. The interesting problem of complementary cycles in strongly connected local tournaments is investigated in Chapter 8. In 1996 Guo and Volkmann [37] solved the question whether a 2-connected local tournament has complementary cycles in characterizing the exceptional digraphs. This result was generalized and extended in various forms. For example, Volkmann and the author [54] characterized all 2-connected in-tournaments that are not cycle complementary and Gould and Guo [32] showed that every $k$-connected local tournament $D$ on at least $5k + 1$ vertices has $k$ vertex disjoint cycles that span the vertex set of $D$. In this chapter we investigate the structure of strongly connected, but not 2-connected, local tournaments that are not cycle complementary. As applications we present sufficient criteria for a strongly connected local tournament $D$ to have $k$ vertex disjoint cycles that span the vertex set of $D$. Our results extend and generalize those of Li and Shu [52, 53] for tournaments. In the last chapter we consider extendable cycles in strongly connected in-tournaments. In 1989 this property was introduced by Hendry [44] in the context of general digraphs and subsequently studied by Tewes and Volkmann [69] for in-tournaments. We prove that every strongly connected in-tournament on $n$ vertices with minimal degree $3 \leq \delta \leq \frac{8n-17}{31}$ is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable which solves a conjecture of Tewes and Volkmann [69] in the affirmative.

Aachen, December 2007 Dirk Meierling

# Contents

# Chapter 1

# Introduction

In this chapter we present the terminology and notation used throughout this thesis. Special notations and definitions will be presented where needed. The results covered in this chapter constitute a collection of basic properties of local tournaments and in-tournaments.

We assume that the reader has a basic knowledge of graph theory and we refer to the books of Volkmann [72, 77] or Bang-Jensen and Gutin [12] for information not given here.

In Section 1.1 we repeat some definitions that are important for this thesis and give specific terminology and notation. Basic structural properties of local tournaments and in-tournaments are introduced in Sections 1.2 and 1.3. The last section covers some simple observations on the order of digraphs.

In the following, any result which is cited from an article or a book is equipped with a heading containing this reference.

## 1.1 Terminology and notation

All digraphs mentioned in this thesis are finite and simple if not noted otherwise. A *digraph* $D$ is an orientation of a *graph*. For a digraph $D$, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of $D$. The numbers $n = n(D) = |V(D)|$ and $m = m(D) = |E(D)|$ are the *order* and *size* of the digraph $D$.

If $D$ is a digraph, a digraph $H$ is called a *subdigraph* of $D$, if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. In this case we also say that $D$ is a *superdigraph* of $H$ and write $H \subseteq D$. If $V(H) = V(D)$, we say that $H$ is a *spanning subdigraph* (or a *factor*) of $D$. The *subdigraph* induced by a subset $A$ of $V(D)$ is denoted by $D[A]$. By $D - A$ we denote the digraph $D[V(D) - A]$. If $A = \{x\}$ is a single vertex, we write $D - x$ instead of $D - \{x\}$. Let $H_1, H_2, \ldots, H_n$ be arbitrary digraphs. Then $D[H_1, H_2, \ldots, H_n]$ denotes the digraph which is obtained by replacing every vertex $x_i$ of $D$ by $H_i$. Furthermore, the arcs between $H_i$ and $H_j$ are defined as follows: $H_i \to H_j$ in $D[H_1, H_2, \ldots, H_n]$ if and only if $x_i \to x_j$ in $D$ and if there is no arc between $x_i$ and $x_j$ in $D$, there is no arc between $H_i$ and $H_j$ in $D[H_1, H_2, \ldots, H_n]$.

If $xy \in E(D)$ is an arc of $D$, the vertex $y$ is a *positive neighbor* or *out-neighbor* of $x$ and $x$ is a *negative neighbor* or *in-neighbor* of $y$. We say that $x$ *dominates* $y$ and $y$ *is dominated by* $x$, denoted by $x \to y$. The vertices $x$ and $y$ are the *end-vertices* of the arc $xy$. Furthermore, we say that two vertices $x$ and $y$ are *adjacent* vertices and that $x$ (or $y$) and $xy$ are *incident*.

Let $A$ and $B$ be two disjoint subdigraphs of a digraph $D$. If every vertex of $A$ dominates every vertex of $B$, we say that $A$ *dominates* $B$ and that $B$ *is dominated by* $A$, denoted by $A \to B$. If there is at least one arc leading from $A$ to $B$ and no arc leading from $B$ to $A$, then $A$ *weakly dominates* $B$ and $B$ *is weakly dominated by* $A$, denoted by $A \rightsquigarrow B$.

By *reversing the arc* $xy$ we mean that we replace the arc $xy$ by the arc $yx$. The *converse* of a directed graph $D$ is the directed graph $D^{-1}$ which we obtain by reversing all arcs of $D$.

The *positive* or *out-neighborhood* $N^+(x)$ of a vertex $x$ is the set of positive neighbors of $x$. More generally, for arbitrary subdigraphs $A$ and $B$ of $D$, the out-neighborhood $N^+(A, B)$ is the set of vertices in $B$ to which there is an arc from a vertex in $A$. The *negative* or *in-neighborhood* $N^-(x)$ of a vertex $x$ and the in-neighborhood $N^-(A, B)$ are defined analogously.

The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *out-degree* and *in-degree* of the vertex $x$, respectively. The *minimum out-degree* $\delta^+(D)$ and the *minimum in-degree* $\delta^-(D)$ of $D$ are given by $\min\{d^+(x) \mid x \in V(D)\}$ and $\min\{d^-(x) \mid x \in V(D)\}$, respectively. Analogously, we define the *maximum out-degree* $\Delta^+(D)$ of $D$ by $\Delta^+(D) = \max\{d^+(x) \mid x \in V(D)\}$ and the *maximum in-degree* $\Delta^-(D)$ by $\Delta^-(D) = \max\{d^-(x) \mid x \in V(D)\}$ of $D$. In addition, let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the *minimum degree* and $\Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ be the *maximum degree* of $D$. We say that a digraph $D$ is *regular* or, more precisely, $\delta(D)$-regular if $\delta(D) = \Delta(D)$.

A *directed path*, or short a *path*, of *length* $\ell$, where $\ell \geq 0$ is an integer, is a subdigraph $P$ of $D$ with $\ell + 1$ distinct vertices $\{v_0, v_1, \ldots, v_\ell\}$ and $\ell$ distinct arcs $\{e_1, e_2, \ldots, e_\ell\}$ such that $e_i = v_{i-1}v_i$ for every $1 \leq i \leq \ell$. For such a path $P$ of length $\ell$ we will use the notation $P = v_0v_1 \ldots v_\ell$. We say that the vertex $v_0$ is the *initial* vertex of $P$, that the vertex $v_\ell$ is the *terminal* vertex of $P$ and that the vertices $v_0$ and $v_\ell$ are *end-vertices* of $P$. Furthermore, we call $P$ a path from $v_0$ to $v_\ell$ or short a *$v_0$-$v_\ell$-path*. More generally, if $A$ and $B$ are two disjoint subdigraphs of $V(D)$ and $P$ is a path such that the initial vertex of $P$ is in $A$ and the terminal vertex of $P$ is in $B$, we say that $P$ is an *$A$-$B$-path*. A *directed cycle*, or short a *cycle*, of *length* $\ell$, where $\ell \geq 3$ is an integer, is a subdigraph $C$ of $D$ with $\ell$ distinct vertices $\{v_1, v_2, \dots, v_\ell\}$ and $\ell$ distinct arcs $\{e_1, e_2, \dots, e_\ell\}$ such that $e_i = v_i v_{i+1}$ for every $1 \leq i \leq \ell - 1$ and $e_\ell = v_\ell v_1$. For such a cycle $C$ of length $\ell$ we will use the notation $C = v_1 v_2 \dots v_\ell v_1$. If $P$ is a path and $x$ is a vertex of $P$, we say that $P$ is a path *through* $x$. Analogously, if $C$ is a cycle and $x$ is a vertex of $C$, we say that $C$ is a cycle *through* $x$. The number of vertices of a path or cycle is also called the *order* of the path or cycle. A cycle or path of order $m$ is an *$m$-cycle* or an *$m$-path*, respectively. If $X$ is a cycle or path of a digraph $D$ with order $|V(D)|$, the subdigraph $X$ is called a *Hamiltonian cycle* or *Hamiltonian path*, respectively. In the first case we call the corresponding digraph $D$ a *Hamiltonian digraph* and in the latter case a *traceable digraph*. An arc $xy$ is called an *$m$-path arc* if the longest path through $xy$ is of order $m$. In the special case $m = n$ the arc is *traceable*. If every arc of a digraph $D$ is traceable, we call $D$ *arc-traceable*. A digraph $D$ is called *pancyclic* if it contains cycles of length $\ell$ for all $\ell \in \{3, 4, \dots, n(D)\}$ and a vertex is called *pancyclic* if it belongs to a cycle of length $\ell$ for every $\ell \in \{3, 4, \dots, n(D)\}$. Furthermore, a digraph $D$ is *vertex pancyclic* if every vertex of $D$ is pancyclic. The *pancyclicity* for arcs is defined analogously. A cycle $C$ is called *extendable* if there is a cycle $C'$ such that $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$. We call a digraph $D$ *cycle complementary* if there exist two vertex disjoint cycles $C_1$ and $C_2$ in $D$ such that $V(D) = V(C_1) \cup V(C_2)$. A *$k$-cycle factor* of a digraph $D$ is a factor $D'$ of $D$ that consists of $k$ vertex disjoint cycles. Let $P = v_1 v_2 \dots v_\ell$ be a path of $D$. Then $P[v_i, v_j]$, where $1 \leq i, j \leq \ell$, denotes the subpath $v_i v_{i+1} \dots v_j$ of $P$ with *initial vertex* $v_i$ and *terminal vertex* $v_j$. The subpaths $C[v_i, v_j]$ of a cycle $C$ are defined analogously. If $x \in V(X)$ for a cycle or path $X$, we denote the *successor* of $x$ on $X$ by $x_X^+$ and the *predecessor* of $x$ on $X$ by $x_X^-$. We also write $x^+$ and $x^-$ if no confusion arises.

A digraph $D$ is *path-mergeable* if for any choice $x, y$ of vertices and any pair $P_1, P_2$ of internally disjoint $x$-$y$-paths, there exists an $x$-$y$-path $Q$ in $D$ such that $V(Q) = V(P_1) \cup V(P_2)$. We call $D$ *in-path-mergeable* if for every vertex $y$ and any pair $P_1, P_2$ of internally disjoint paths ending in $y$, there is a path $Q$ with the following three properties; the path $Q$ consists of all vertices of $P_1$ and $P_2$, it starts at a vertex which is the initial vertex of either $P_1$ or $P_2$ and it terminates at $y$. Note that, in this definition, the initial vertices of $P_1$ and $P_2$ may coincide. Hence, every in-path-mergeable digraph is path-mergeable, whereas the converse is not necessarily true.

A digraph is *acyclic* if it has no cycles. Let $D$ be a digraph and let $x_1, x_2, \ldots, x_n$ be an ordering of its vertices. We call this ordering an *acyclic ordering* if for every arc $x_ix_j$ in $D$, we have $i < j$.

We speak of a *connected digraph* if the underlying graph is connected. A digraph $D$ is said to be *strongly connected* or just *strong*, if for every pair $x, y$ of vertices of $D$, there is a path from $x$ to $y$. A *strong component* of $D$ is a maximal induced strong subdigraph of $D$.

The *condensation* or *strong component digraph* $SC(D)$ of a digraph $D$ is obtained by contracting the strong components of $D$ into vertices and deleting any parallel arcs obtained in this process. In other words, if $D_1, D_2, \ldots, D_p$, where $p \geq 1$, are the strong components of $D$, the vertex set of $SC(D)$ is $V(SC(D)) = \{v_1, v_2, \ldots, v_p\}$ and the arc set of $SC(D)$ is $E(SC(D)) = \{v_iv_j \mid N^+(D_i, D_j) \neq \emptyset\}$. Note that $SC(D)$ is acyclic and thus has an acyclic ordering, that is the strong components of $D$ can be labeled $D_1, D_2, \ldots, D_p$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$. We call this ordering the *acyclic ordering of the strong components* of $D$. The strong components corresponding to vertices with in-degree (out-degree) zero in $SC(D)$ are called *initial (terminal) strong components* of $D$.

We say that a digraph $D$ is *$k$-strongly connected* or just *$k$-strong* if $D$ has at least $k+1$ vertices and for any set $S$ of vertices such that $|S| < k$, the subdigraph $D-S$ is strong. A set $S$ of vertices is called a *separating set* if $D - S$ is not strong. We speak of a *minimal separating set* (*minimum separating set*) $S$ of a digraph $D$ if $D - S$ is not strong and there exists no separating set $S' \subseteq S$ with $|S'| < |S|$ (no separating set $S''$ with $|S''| < |S|$). If $S = \{x\}$ for a separating set $S$, we call $x$ a *separating vertex* or *cut-vertex*. Analogously, a vertex $x$ is called a *non-separating vertex* of a strong digraph $D$ if $D - x$ is strong.

An *in-branching* of a digraph $D$ is a spanning tree of $D$ rooted at some vertex $v$ and oriented in such a way that every vertex other than $v$ has an arc out of it. An *out-branching* is defined analogously.

A digraph is *semicomplete* if for any two different vertices $x$ and $y$, there is at least one arc between them. A *tournament* is a semicomplete digraph without 2-cycles. We speak of a *transitive* tournament $T$ if there exists no cycle in $T$. Note that in this case a tournament $T$ has a unique Hamiltonian path.

Throughout this thesis we only consider connected digraphs and all subscripts are taken modulo the corresponding number.

The topic of this thesis is the investigation of two classes of digraphs that are generalizations of tournaments: *local tournaments* and *in-tournaments*. In 1990, Bang-Jensen [3] defined *local tournaments* to be the family of oriented graphs

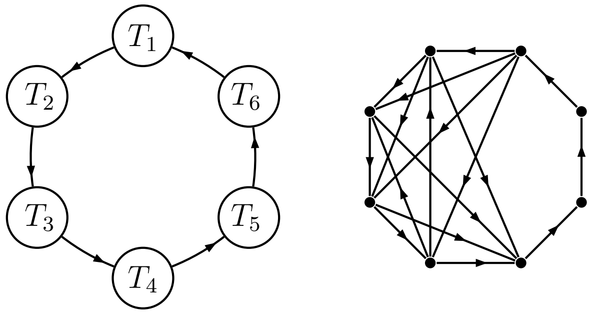


Figure 1.1: Local tournaments.

where the in-neighborhood as well as the out-neighborhood of every vertex induces a tournament.

**Definition 1.1.** *A* local tournament *is a digraph such that the in-neighborhood as well as the out-neighborhood of every vertex induces a tournament.*

In transferring the general adjacency only to vertices that have a common positive or a common negative neighbor, local tournaments are an interesting generalization of tournaments. We note that every induced subdigraph of $D$ is still a local tournament and, in addition, the converse $D^{-1}$ of $D$ is also a local tournament. The digraphs in Figure 1.1 show that the class of local tournaments is more general than the class of tournaments. Here $T_i$ is a strongly connected tournament such that $T_i$ dominates $T_{i+1}$ for $i = 1, 2, \ldots, 6$.

Ever since the initial article of Bang-Jensen [3] was published, a lot of research has been done concerning local tournaments, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length two. In particular, the Ph.D. theses of Guo [33] and Huang [45] handled this subject in detail. For more information concerning different generalizations of tournaments, the reader may be referred to the survey article of Bang-Jensen and Gutin [11].

In 1993, Bang-Jensen, Huang and Prisner [16] introduced a further generalization of local tournaments, the class of *in-tournaments*, in claiming adjacency only for vertices that have a common positive neighbor.

**Definition 1.2.** *An* in-tournament *is a digraph such that the in-neighborhood of every vertex induces a tournament.*

It is clear that every local tournament is also an in-tournament. In addition, every induced subdigraph of an in-tournament is also an in-tournament. The digraphs in Figure 1.2 show that the class of in-tournaments is more general than the class

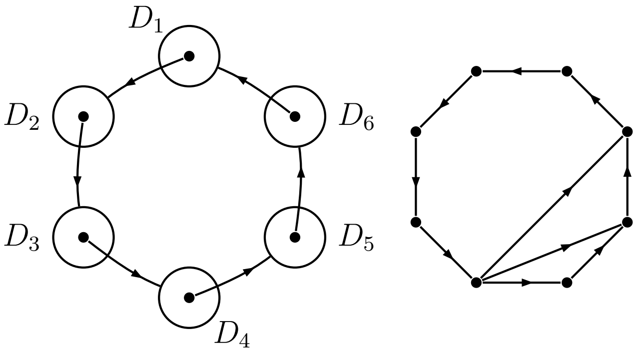


Figure 1.2: In-tournaments.

of local tournaments. Here $D_i$ is a strongly connected in-tournament such that $D_i$ weakly dominates $D_{i+1}$ for $i = 1, 2, \ldots, 6$.

Initially, some problems concerning in-tournaments were studied by Bang-Jensen, Huang and Prisner [16]. Later on, Tewes and Volkmann [68, 69], Volkmann [74] and Tewes [66, 67] focused on the cycle and path structure of this class of digraphs.

To make our presentation more complete, we will briefly introduce two other interesting generalizations of tournaments.

A *multipartite tournament* is an orientation of a complete multipartite graph. Hence a multipartite tournament is a tournament if and only if every partite set has size one. For information on multipartite tournaments we refer the reader to the comprehensive survey articles of Gutin [40] and Volkmann [75, 78].

In a *quasi-transitive digraph* $D$ every vertex that belongs to the negative neighborhood of an arbitrary vertex $x \in V(D)$ is adjacent to every vertex of the positive neighborhood of $x$. As in the definitions of local tournaments and in-tournaments the general adjacency of vertices in tournaments is transferred only to vertices which share a local property. Results on quasi-transitive digraphs can be found in [6, 14, 15, 17, 30].

In the next sections we shall introduce the basic properties of local tournaments and in-tournaments. These results play an important role in this thesis.

## 1.2 Basic properties of local tournaments

In this section we introduce the basic properties of local tournaments.

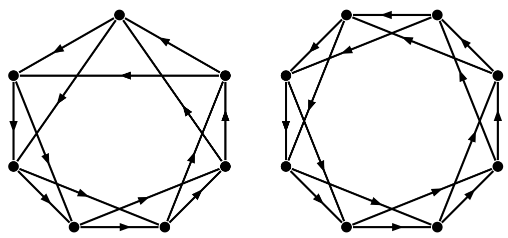

Figure 1.3: Members of the class $\mathcal{R}_n^2$.

### 1.2.1 Non-strong local tournaments

The following class of digraphs plays an important role in the study of local tournaments.

**Definition 1.3.** *A digraph on $n$ vertices is called a* round digraph *if its vertices $v_1, v_2, \ldots, v_n$ can be labelled such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-d^-(v_i)}\}$ for every $i$, where the subscripts are taken modulo $n$. We refer to $v_1, v_2, \ldots, v_n$ as a* round labeling *of $D$.*

**Definition 1.4.** *A local tournament $D$ is* round-decomposable *if there exists a round local tournament $R$ on $r \geq 2$ vertices and strong local subtournaments $H_1, H_2, \ldots, H_r$ of $D$ such that $D = R[H_1, H_2, \ldots, H_r]$. We call $R[H_1, H_2, \ldots, H_r]$ a* round decomposition *of $D$.*

From these definitions it is easy to see that every strongly connected, round-decomposable local tournament has a Hamiltonian cycle. For example, the first digraph in Figure 1.1 is round-decomposable and it has the round decomposition $R[T_1, T_2, \ldots, T_6]$, where $R$ is a cycle of length six. A special class of round local tournaments is introduced in the next definition.

**Definition 1.5.** *Let $\mathcal{R}_n^2$ be the set of all round local tournaments that are 2-regular.*

The next theorem shows that every connected, but not strongly connected local tournament is round-decomposable.

**Theorem 1.6** (Bang-Jensen [3] 1990)**.** *Let $D$ be a connected local tournament.*

*(a) If $A$ and $B$ are two strong components of $D$, then either there is no arc between them or $A$ dominates $B$ or $B$ dominates $A$;*

*(b) If $A$ and $B$ are two strong components of $D$ such that $A$ dominates $B$, then $D[A]$ and $D[B]$ are tournaments;*

*(c) The strong components of $D$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$, where $p \geq 1$, such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i$ dominates $D_{i+1}$ for $i = 1, 2, \ldots, p-1$.*

According to Theorem 1.6, we give the following definition.

**Definition 1.7.** *Let $D$ be a connected local tournament. Then the unique sequence $D_1, D_2, \ldots, D_p$ as defined in Theorem 1.6 is called the* strong decomposition *of $D$. Furthermore, we call $D_1$ the* initial strong component *and $D_p$ the* terminal strong component *of $D$.*

As immediate corollaries from Theorem 1.6 we get the following results.

**Corollary 1.8.** *Every connected, but not strongly connected, local tournament $D$ has a unique round decomposition $R[D_1, D_2, \ldots, D_p]$, where $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D$ and $R$ is a round acyclic local tournament.*

**Corollary 1.9.** *If $D$ is a connected local tournament, then $D$ has a Hamiltonian path.*

From the fact that every connected non-strong local tournament has a unique strong decomposition, Guo and Volkmann [36] found a further useful decomposition, which is formulated in the next theorem (cf. Figure 1.4).

**Theorem 1.10** (Guo & Volkmann [36] 1994)**.** *Let $D$ be a connected local tournament that is not strong. Then $D$ can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, \ldots, D'_r$ such that every $D'_i$ is a tournament and the following holds.*

*(a) $D'_1$ is the terminal component of $D$ and $D'_i$ consists of some strong components of $D$ for $i \geq 2$;*

*(b) $D'_{i+1}$ dominates the initial strong component of $D'_i$ and there exists no arc from $D'_i$ to $D'_{i+1}$ for $i = 1, 2, \ldots, r-1$;*

*(c) If $r \geq 3$, then there is no arc between $D'_i$ and $D'_j$ for $i$ and $j$ satisfying $|j-i| \geq 2$.*

According to Theorem 1.10, we give the following definition.

**Definition 1.11.** *Let $D$ be a connected local tournament that is not strong. Then the unique sequence $D'_1, D'_2, \ldots, D'_r$ as defined in Theorem 1.10 is called the* semicomplete decomposition *of $D$.*

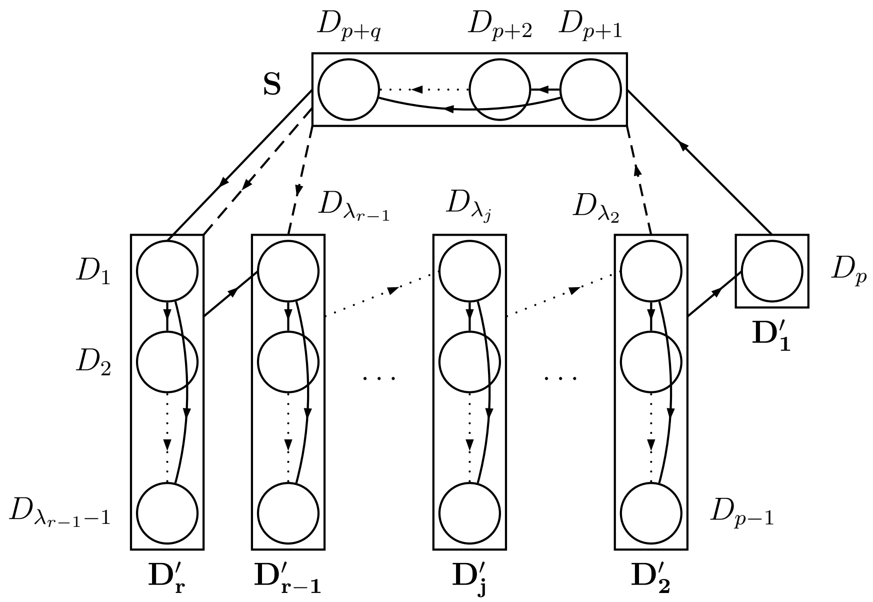


Figure 1.4: The semicomplete decomposition of a local tournament.

### 1.2.2 Strong local tournaments

In this subsection the structure of strong local tournaments is established. The first result is a useful observation about the interaction of a cycle and an external vertex.

**Lemma 1.12** (Bang-Jensen [3] 1990). *Let $D$ be a local tournament containing a cycle $C = u_1u_2 \dots u_ku_1$. If there exists a vertex $v \in V(D) - V(C)$ such that $N^+(v, C) \neq \emptyset$ (or $N^-(C, v) \neq \emptyset$), then either $v \to C$ ($C \to v$, respectively) or $u_i \to v \to u_{i+1}$ for some integer $1 \leq i \leq k$, i.e. there exists a cycle $C'$ in $D$ such that $V(C') = V(C) \cup \{v\}$.*

As a consequence we can generalize the well-known result of Camion [27] that a tournament is Hamiltonian if and only if it is strong to local tournaments.

**Theorem 1.13** (Bang-Jensen [3] 1990). *A local tournament is strong if and only if it has a Hamiltonian cycle.*

The following observation is the first step to the classification of connected local tournaments (cf. Theorem 1.18).

**Lemma 1.14** (Bang-Jensen [3] 1990). *If $D$ is a strong local tournament and $S$ is a minimal separating set of $D$, then $D - S$ is connected.*

The next result is a characterization of local tournaments that are both strongly connected and round-decomposable.

**Theorem 1.15** (Bang-Jensen, Guo, Gutin & Volkmann [8] 1997)**.** *Let $D$ be a strong local tournament and let $S$ be a minimal separating set of $D$. Then $D$ is round-decomposable if and only if $D = R[D_1, D_2, \ldots, D_{p+q}]$, where $R$ is a round local tournament, $D_i$ is a strong tournament for every $i$, $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - S$, where $p \geq 2$, and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ is the strong decomposition of $D[S]$, where $q \geq 1$.*

Together with Corollary 1.8 the above theorem yields the following result.

**Corollary 1.16** (Bang-Jensen, Guo, Gutin & Volkmann [8] 1997)**.** *Let $D$ be a round-decomposable local tournament. Then $D$ has a* unique *round decomposition $D = R[D_1, D_2, \ldots, D_\alpha]$, where $D_i$ is a strong tournament for every $i = 1, 2, \ldots, \alpha$.*

Now we give a characterization of strong local tournaments that are neither tournaments nor round-decomposable.

**Theorem 1.17** (Bang-Jensen, Guo, Gutin & Volkmann [8] 1997)**.** *Let $D$ be a connected local tournament which is not a tournament. Then $D$ is not round-decomposable if and only if the following conditions are satisfied:*

*(a) There is a minimal separating set $S$ of $D$ such that $D - S$ is not a tournament and for each such $S$ the digraph $D[S]$ is a tournament and the semicomplete decomposition of $D - S$ has exactly three components $D'_1$, $D'_2$ and $D'_3$;*

*(b) There are integers $\alpha$, $\beta$, $\mu$ and $\nu$ with $\lambda \leq \alpha \leq \beta \leq p - 1$ and $p + 1 \leq \mu \leq \nu \leq p + q$ such that*

$$N^-(D_\alpha, D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha, D_\nu) \neq \emptyset$$
$$\text{or } N^-(D_\mu, D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu, D_\beta) \neq \emptyset,$$

*where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and $D_\lambda$ is the initial component of $D'_2$.*

As the final result in this subsection we give the following theorem which shows that every connected local tournament belongs to one of three well-described classes.

**Theorem 1.18** (Bang-Jensen, Guo, Gutin & Volkmann [8] 1997)**.** *Let $D$ be a connected local tournament. Then exactly one of the following possibilities holds.*

*(a) $D$ is round-decomposable and there is a local tournament $R$ on $\alpha \geq 3$ vertices such that $R[D_1, D_2, \ldots, D_\alpha]$ is the unique round decomposition of $D$, where $D_i$ is a strong tournament for $i = 1, 2, \ldots, \alpha$;*

*(b) D is not round-decomposable and not a tournament and it has the structure as described in Theorem 1.17;*

*(c) D is a tournament that is not round-decomposable.*

## 1.3 Basic properties of in-tournaments

In this section we introduce the basic properties of in-tournaments.

### 1.3.1 Non-strong in-tournaments

The first results deal with very useful properties of in-tournaments.

**Theorem 1.19** (Bang-Jensen [5] 1995). *If $D$ is an in-tournament, then $D$ is in-path-mergeable.*

**Corollary 1.20** (Bang-Jensen [5] 1995). *If $D$ is an in-tournament, then $D$ is path-mergeable.*

**Corollary 1.21.** *Every local tournament is in-path-mergeable and path-mergeable.*

In Section 1.2 it was shown that the structure of local tournaments is very similar to the structure of tournaments: any strong component is a tournament, if two strong components are adjacent, then one completely dominates the other and finally the condensation $SC(D)$ has a unique Hamiltonian path. In-tournaments and (local) tournaments do not have all of these structural properties in common as we shall see in the following. The first result is an analog to Theorem 1.6.

**Theorem 1.22** (Bang-Jensen, Huang & Prisner [16] 1993). *Let $D$ be an in-tournament.*

*(a) If $A$ and $B$ are two distinct strong components of $D$, either there is no arc between them or $A$ weakly dominates $B$ or $B$ weakly dominates $A$. Furthermore, if $A$ weakly dominates $B$, the set $N^-(B, A)$ dominates $B$.*

*(b) If $A$ and $B$ are two distinct strong components of $D$ such that $A$ weakly dominates $B$, the set $N^-(b, A)$ induces a tournament for every $b \in B$.*

*(c) If $D$ is connected, then $D$ has an out-branching $SC(D)$. If $R$ is the root of $SC(D)$ and $A$ is an arbitrary other strong component of $D$, there is a path from $R$ to $A$ containing all components that can reach $A$.*

Since in-tournaments are in-path-mergeable, we obtain a characterization for in-tournaments that have a Hamiltonian path.

**Theorem 1.23** (Bang-Jensen, Huang & Prisner [16] 1993)**.** *A connected in-tournament has a Hamiltonian path if and only if it has an in-branching.*

In addition we note the following.

**Remark 1.24.** *Let $D$ be a connected in-tournament and let $SC(D)$ be the condensation of $D$. The strong components of $D$ can be ordered in a way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$. In particular $N^-(D_1) - V(D_1) = N^+(D_p) - V(D_p) = \emptyset$.*

### 1.3.2 Strong in-tournaments

The first result is a useful observation about the interaction of a cycle and a vertex which lies outside.

**Theorem 1.25** (Bang-Jensen, Huang & Prisner [16] 1993)**.** *Let $D$ be an in-tournament and let $C = u_1u_2 \ldots u_su_1$ be a cycle in $D$. If $|N^+(x, C)| \geq 1$ for a vertex $x \notin V(C)$, then either $x \to C$ or $u_i \to x \to u_{i+1}$ for an integer $1 \leq i \leq s$.*

In 1993, Bang-Jensen, Huang and Prisner [16] generalized the well known result that a tournament is strong if and only if it is Hamiltonian to in-tournaments. We can also deduce this result from the above theorem.

**Theorem 1.26** (Bang-Jensen, Huang & Prisner [16] 1993)**.** *An in-tournament is strong if and only if it has a Hamiltonian cycle.*

More of the structure of local tournaments is preserved in strongly connected in-tournaments as we see in the next theorem.

**Theorem 1.27** (Bang-Jensen, Huang & Prisner [16] 1993)**.** *Let $D$ be a strong in-tournament and let $S$ be a minimal separating set of $D$. The strong components of $D - S$ can be ordered in a unique way $D_1$, $D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i$ has an arc to $D_{i+1}$ for $i = 1, 2, \ldots, p-1$.*

According to Theorem 1.27 we give the following definition.

**Definition 1.28.** *The unique labeling $D_1, D_2, \ldots, D_p$ of the strong components of $D - S$ as described in Theorem 1.27 is called* the strong decomposition of $D - S$. *We call $D_1$ the* initial *and $D_p$ the* terminal component.

As a consequence of Theorem 1.27, Bang-Jensen, Huang and Prisner obtained the following structural result.

**Corollary 1.29** (Bang-Jensen, Huang & Prisner [16] 1993)**.** *Let $D$ be a strong in-tournament and let $S$ be a minimal separating set of $D$. The strong decomposition $D_1, D_2, \ldots, D_p$, where $p \geq 2$, of $D - S$ has the following properties.*

*(a) If $x_i \rightarrow x_k$ for $x_i \in V(D_i)$ and $x_k \in V(D_k)$ with $1 \leq i \neq k \leq p$, then $x_i \rightarrow D_j$ for every $i + 1 \leq j \leq k$.*

*(b) The digraph $D - S$ has a Hamiltonian path.*

*(c) For every $s \in S$ we have $d^+(s, D_1) > 0$ and $d^-(s, D_p) > 0$.*

At the end of this section we show how strong in-tournaments can be decomposed in larger components similarly to Theorem 1.10.

**Theorem 1.30** (Meierling & Volkmann [54])**.** *Let $D$ be a strong in-tournament and let $S$ be a minimal separating set of $D$. There is a unique order $D_1^{'}, D_2^{'}, \ldots, D_r^{'}$ with $r \geq 2$ of the strong components of $D - S$ such that*

*(a) $D_1'$ is the terminal component of $D - S$ and $D_i'$ consists of some strong components of $D$ for $i \geq 2$;*

*(b) there exists a vertex $x$ in the initial component of $D_{i+1}'$ and a vertex $y$ in the terminal component of $D_{i+1}'$ such that $\{x, y\}$ dominates the initial component of $D_i'$ for $i = 1, 2, \ldots, r - 1$;*

*(c) there are no arcs between $D_i'$ and $D_j'$ for $i, j$ satisfying $|i - j| \geq 2$;*

*(d) if $r \geq 3$, there exist no arcs from $D_i'$ to $S$ for $i \geq 3$, $S \rightarrow D_1$ and $S$ induces a tournament in $D$.*

*Proof.* Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - S$. We define (cf. Figure 1.5)

$$D_1^{'} = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\left\{ j \mid N^+(D_j, D_i^{'}) \neq \emptyset \right\}$$

and

$$D_{i+1}^{'} = D\left[V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \ldots \cup V(D_{\lambda_i - 1})\right].$$

So we have a new decomposition $D_1', D_2', \ldots, D_r'$ of $D$, where $2 \leq r \leq p$, that satisfies (a).

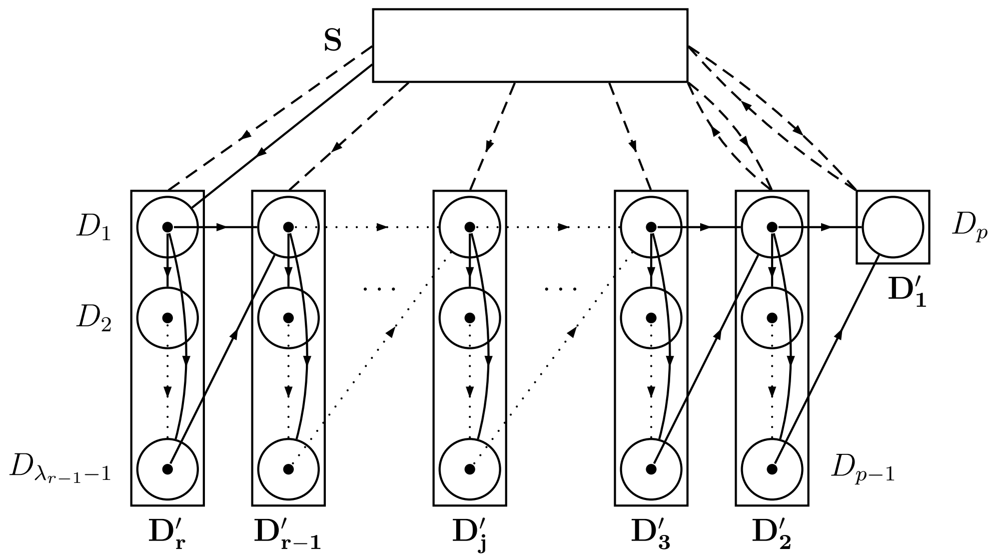


Figure 1.5: The decomposition of a strong in-tournament.

By the definition of $D'_{i+1}$, there exists a strong component $D_l$ of $D'_i$ such that $N^+(D_{\lambda_{i+1}}, D_\ell) \neq \emptyset$. Therefore we conclude from Theorem 1.27 and Corollary 1.29 that there exists a vertex $x \in V(D_{\lambda_{i+1}})$ such that $x \to D_j$ for every index $j \in \{\lambda_{i+1}, \lambda_{i+1}+1, \ldots, \ell\}$. From Corollary 1.29 it follows that there exists a vertex $y \in V(D_{\lambda_i - 1})$ such that $y \to D_{\lambda_i}$. So (b) has been proved.

Note that if $r = 2$, there is nothing to prove in (c). If $r \geq 3$ and $i, j$ are two integers with $i \geq j+2$, there is no arc from $D'_i$ to $D'_j$ by the definition of $\lambda_{i-1}$. In addition, $D$ contains no arc from $D'_j$ to $D'_i$ by Theorem 1.27.

Assume to the contrary that there is an arc $xs$ from $x \in V(D'_i)$ to $s \in S$, where $i \geq 3$. Note that Corollary 1.29.(c) states that $s$ has a negative neighbor $x'$ in $D_p$. Since $D$ is an in-tournament, it follows that $x$ and $x'$ are adjacent, a contradiction to (c).

Now we shall prove that $S \to D_1$. Note that we have $d^+(s, D_1) > 0$ for every vertex $s \in S$ by Corollary 1.29.(c). Now let $s \in S$ be an arbitrary vertex. If $D_1$ consists of a single vertex, there is nothing to prove. Otherwise $D_1$ has a Hamiltonian cycle by Theorem 1.26. Using Theorem 1.25, we deduce that either $s \to D_1$ or that $s$ has a negative neighbor in $D_1$. Thus, if $s \not\to D_1$, the vertex $s$ has negative neighbors both in $D_1$ and $D_p$, a contradiction to (c). This completes the proof of this theorem. □

## 1.4 Observations on the order of digraphs

In this section we present some simple lower bounds for the order of digraphs which will be frequently used in this thesis.

**Lemma 1.31.** *Let $D$ be a digraph of order $n$ without cycles of length two and let $r \geq 1$ be an integer.*

*(a) If $d^-(x) + d^+(y) \geq r$ for every arc $xy$ of $D$, then $n \geq r + 1$;*

*(b) If $\delta^+(D) \geq r$ or $\delta^-(D) \geq r$, then $n \geq 2r + 1$;*

*(c) If $\delta^+(D) \geq r$ and $\Delta^+(D) \geq r + 1$ or if $\delta^-(D) \geq r$ and $\Delta^-(D) \geq r + 1$, then $n \geq 2r + 2$.*

*Proof.* (a) Observe that

$$\begin{aligned} r\frac{n(n-1)}{2} \leq \sum_{xy\in E(D)} (d^-(x) + d^+(y)) &= \sum_{xy\in E(D)} d^-(x) + \sum_{xy\in E(D)} d^+(y) \\ &= 2\sum_{x\in V(D)} d^-(x)d^+(x) \\ &\leq 2n(\frac{n-1}{2})^2. \end{aligned}$$

So

$$r\frac{n(n-1)}{2} \leq 2n(\frac{n-1}{2})^2$$

which finally implies that $n \geq r + 1$.

(b) Suppose, without loss of generality, that $\delta^+(D) \geq r$. It follows that

$$\frac{n(n-1)}{2} \geq \sum_{x\in V(D)} d^+(x) \geq nr$$

which finally implies that $n \geq 2r + 1$.

(c) Suppose, without loss of generality, that $\delta^+(D) \geq r$ and that there exists a vertex $x \in V(D)$ such that $d^+(x) \geq r + 1$. It follows that

$$\frac{n(n-1)}{2} \geq \sum_{x\in V(D)} d^+(x) \geq r + 1 + (n-1)r = nr + 1.$$

Since $n$ is an integer, it follows that $n \geq 2r + 2$. □

# Part I

# Paths in local tournaments and in-tournaments

# Chapter 2

# Arc-traceable local tournaments

A digraph $D$ is *arc-traceable* if for every arc $xy$ of $D$, the arc $xy$ belongs to a Hamiltonian path of $D$. The first result concerning Hamiltonian paths in digraphs is due to Rédei [60] who showed already in 1934 that every tournament contains a Hamiltonian path.

**Theorem 2.1** (Rédei [60] 1934)**.** *Every tournament contains a Hamiltonian path.*

In 1990, Bang-Jensen [3] showed the same result for connected local tournaments (cf. Corollary 1.9). A first sufficient condition for a digraph to contain a Hamiltonian cycle (and thus, in particular a Hamiltonian path) was given in 1960.

**Theorem 2.2** (Ghouila-Houri [31] 1960)**.** *If $D$ is a digraph such that $\delta(D) \geq |V(D)|/2$, then $D$ contains a Hamiltonian cycle.*

Moon [57] proved that in a strongly connected tournament every vertex belongs to cycles of arbitrary lengths.

**Theorem 2.3** (Moon [57] 1966)**.** *A tournament $T$ is strong if and only if $T$ is vertex pancyclic.*

Weaker results were shown by Camion [27] in 1959 and by Harary and Moser [42] in 1966.

**Theorem 2.4** (Camion [27] 1959)**.** *A tournament $T$ is strong if and only if $T$ is Hamiltonian.*

**Theorem 2.5** (Harary & Moser [42] 1966)**.** *A tournament $T$ is strong if and only if $T$ is pancyclic.*

Since tournaments are known to have a Hamiltonian path and strongly connected tournaments have Hamiltonian cycles, it is a natural question to ask under which conditions every arc of a given tournament is part of a Hamiltonian path or cycle. In fact, the condition that every arc of a tournament belongs to a cycle of length $k$ for every integer $3 \leq k \leq n$ was introduced as *arc-pancyclicity* and the following results were given.

**Theorem 2.6** (Alspach [1] 1967)**.** *Every regular tournament is arc-pancyclic.*

**Theorem 2.7** (Jakobsen [48] 1972)**.** *Every arc of an almost-regular $n$-tournament, where $n \geq 8$, is contained in a cycle of length $k$ for every $4 \leq k \leq n$.*

**Theorem 2.8** (Thomassen [70] 1980)**.** *If $T$ is an $n$-tournament such that $\Delta(T) - \delta(T) \leq \frac{n-3}{5}$, then every arc of $T$ is contained in a cycle of length $k$ for every $4 \leq k \leq n$.*

In contrast to arc-pancyclicity the question of arc-traceability was not addressed. In 2006, inspired by Volkmann's article [76], Busch, Jacobson and Reid [26] studied the structure of tournaments that are not arc-traceable and consequently gave various sufficient conditions for tournaments to be arc-traceable. Inspired by the article of Busch, Jacobson and Reid, we develop in this chapter the structure necessary for a local tournament to be not arc-traceable. Using this structure, we give sufficient conditions for a local tournament to be arc-traceable and we present examples showing that these conditions are best possible. Busch, Jacobson and Reid proved the following result (cf. Figure 2.1).

**Theorem 2.9** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a strong tournament with an arc $xy \in E(T)$ that is not on a Hamiltonian path, then*

*(a) there exists a vertex $z$ such that $T - z$ is not strong;*

*(b) $T - z$ has $p \geq 4$ strong components;*

*(c) $x$ is in the initial strong component of $T - z$ and $y$ is in the terminal strong component of $T - z$;*

*(d) $z$ is dominated by the $2^{nd}$ strong component of $T - z$ and $z$ dominates the $(p-1)^{st}$ strong component of $T - z$.*

Using the structural result above, they proved the following theorems.

**Theorem 2.10** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a strong $n$-tournament with $\delta \geq 2$ such that $d^-(x) + d^+(y) \geq \frac{n}{2} - 2$ for every arc $xy \in E(T)$, then $T$ is arc-traceable.*

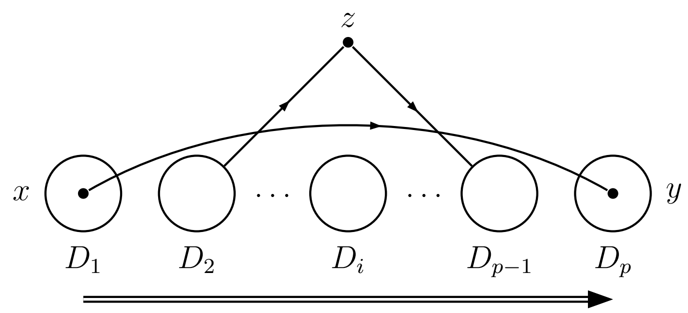


Figure 2.1: The structure of a tournament with the non-traceable arc $xy$.

**Theorem 2.11** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament and let $xy \in E(T)$ be an arc of $T$ such that there exist at least two (internally) vertex disjoint paths from $y$ to $x$ in $T$. Then there exists a Hamiltonian path of $T$ which includes the arc $xy$.*

For strongly connected, but not 2-connected local tournaments, Volkmann [74] showed the following.

**Theorem 2.12** (Volkmann [74] 2000)**.** *Let $u$ be a vertex of a strong local tournament such that $D - u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, then the arcs from $D_i$ to $D_{i+1}$ for $1 \leq i \leq p - 1$ and the arcs in $D_i$ for $2 \leq i \leq p - 1$ are contained in a Hamiltonian path of $D$.*

In this chapter, we transfer Theorems 2.9, 2.10 and 2.11 to the class of local tournaments.

## 2.1 Sufficient criteria for arc-traceability

In this section we give various criteria for a strong local tournament to be arc-traceable.

### 2.1.1 Sufficient criteria for arc-traceability in terms of (local) connectivity

We begin with local tournaments that are not strongly connected.

**Observation 2.13.** *Let $D$ be a connected, but not strongly connected local tournament with the strong decomposition $D_1, D_2, \ldots, D_p$, where $p \geq 2$. Then $D$ is* *arc-traceable if and only if each of the strong components is arc-traceable and there exists no arc leading from $D_i$ to $D_j$ for $j \geq i+2$.*

As a consequence of this observation, we can now focus on strongly connected local tournaments. For in-tournaments (and thus, in particular for local tournaments and tournaments) we can show the following result.

**Observation 2.14** (Volkmann [74] 2000)**.** *Let $uv$ be an arbitrary arc of a strong in-tournament $D$. If $D-u$ or $D-v$ is strong, then $D$ contains a Hamiltonian path starting with the arc $uv$ or ending with the arc $uv$, respectively.*

*Proof.* Let $D$ be a strong in-tournament and let $x \in V(D)$ be a vertex such that $D-x$ is strong. According to Theorem 1.26, the digraph $D-x$ has a Hamiltonian cycle $C = v_1v_2\ldots v_{n-1}v_1$. Hence each arc $xv_i$ is on the Hamiltonian path $xC[v_i, v_{i-1}]$ as well as each arc $v_jx$ belongs to the Hamiltonian path $C[v_{j+1}, v_j]x$. □

As immediate consequences we obtain the following corollaries.

**Corollary 2.15.** *If $D$ is a 2-connected in-tournament, then $D$ is arc-traceable.*

**Corollary 2.16** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a strong tournament and $u$ is not a separating vertex of $T$, then every arc incident with $u$ belongs to a Hamiltonian path of $T$.*

**Corollary 2.17** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a 2-connected tournament, then $T$ is arc-traceable.*

Using Theorem 1.27, we can show the following proposition.

**Theorem 2.18.** *If $D$ is a strong in-tournament with the property that there exist at least two internally vertex disjoint paths from $y$ to $x$ for every arc $xy$, then $D$ is arc-traceable.*

*Proof.* Let $x$ be an arbitrary vertex of $D$. By Observation 2.14 it suffices to show that $x$ is not a separating vertex of $D$. So assume that $D-x$ is not strong and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-x$, where $p \geq 2$. Let $uv$ be an arc of $D$ such that $u \in V(D_1)$ and $v \in V(D_2)$. Then all paths from $v$ to $u$ include the vertex $x$, a contradiction to our assumption. □

In particular we derive the following result.

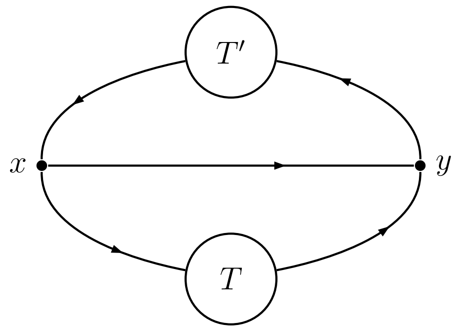


Figure 2.2: A class of local tournaments with two internally vertex disjoint paths from $y$ to $x$ such that there is no Hamiltonian path through $xy$.

**Corollary 2.19** (Busch [25] 2005)**.** *If $T$ is a tournament with the property that there exist at least two internally vertex disjoint paths from $y$ to $x$ for every arc $xy$, then $T$ is arc-traceable.*

Busch, Jacobson and Reid [26] showed that for tournaments this property applies locally as well.

**Theorem 2.20** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament and let $xy \in E(T)$ be an arc of $T$ such that there exist two internally vertex disjoint paths from $y$ to $x$ in $T$. Then there exists a Hamiltonian path through $xy$ in $T$.*

However, the following example shows that Theorem 2.20 is no longer valid for local tournaments (cf. Figure 2.2).

**Example 2.21.** *Let $T$ be an arbitrary tournament and let $T'$ be a tournament on at least two vertices. We define the local tournament $D$ by the vertex set*

$$V(D) = V(T) \cup V(T') \cup \{x, y\}$$

*and the edge set*

$$\begin{aligned} E(D) = &E(T) \cup E(T') \cup \{xy\} \cup \{xv \mid v \in V(T)\} \cup \{vy \mid v \in V(T)\} \cup \\ &\{wx \mid w \in V(T')\} \cup \{yw \mid w \in V(T')\}. \end{aligned}$$

*Then there exist two internally vertex disjoint paths from $y$ to $x$ in $D$, but there is no Hamiltonian path through $xy$ in $D$.*

Note that every local tournament $D$ as defined in Example 2.21 has the following two properties. Firstly, the vertices $x$ and $y$ are the only separating vertices of $D$ and, secondly, there is no arc between the internal vertices of every longest path $P$ from $y$ to $x$ and $D - P$. The next results show that these properties are indeed necessary for an arc $xy$ to be not traceable.

**Theorem 2.22.** *Let $D$ be a strong local tournament with at least three separating vertices and let $xy$ be an arc of $D$ such that there exist at least two (internally) vertex disjoint paths from $y$ to $x$ in $D$. Then there exists a Hamiltonian path of $D$ that includes the arc $xy$.*

*Proof.* Let $z \notin \{x, y\}$ be a separating vertex of $D$. Since there are at least two (internally) vertex disjoint paths from $y$ to $x$ in $D$, the vertices $x$ and $y$ are in the same strong component of $D-z$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-z$, where $p \geq 2$. According to Theorem 2.12, every arc in $D_i$ for $2 \leq i \leq p-1$ belongs to a Hamiltonian path of $D$. Hence we may assume that $xy \in E(D_i)$ for an index $i \in \{1, p\}$. By symmetry we may assume, without loss of generality, that $i = 1$.

*Case 1:* Suppose that there exists only one (internally) vertex disjoint path from $y$ to $x$ in $D_1$. Then every pair $P_1$, $P_2$ of (internally) vertex disjoint paths from $y$ to $x$ in $D$ has the property that $z \in V(P_1)$ and $V(P_2) \subseteq V(D_1)$ or that $z \in V(P_2)$ and $V(P_1) \subseteq V(D_1)$ . We may assume, without loss of generality, that the latter assumption holds. Assume now that we have chosen a pair $P_1, P_2$ of (internally) vertex disjoint path from $y$ to $x$ in $D$ under the following conditions:

- **A.** the successor of $y$ on $P_2$ is not in $D_1$,
- **B.** under condition A: $|V(P_1)| + |V(P_2)|$ is maximal.

It follows by condition B that $V(D) - V(D_1) \subseteq V(P_2)$. Let $R = V(D) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices. Since $R$ induces a tournament in $D$, there exists a Hamiltonian path $Q$ of $D[R]$ (cf. Theorem 2.1). Therefore $D$ contains the Hamiltonian path

$$QP_2[y^+, x]P_1[y, x^-]$$

through $xy$.

*Case 2:* Suppose that there exist two (internally) vertex disjoint paths from $y$ to $x$ in $D_1$. Let $P_1 = yu_1u_2 \ldots u_sx$ and $P_2 = yw_1w_2 \ldots w_tx$ be two such paths such that

$$|V(P_1)| + |V(P_2)| \geq |V(P_1')| + |V(P_2')|$$

for any pair $P_1'$, $P_2'$ of (internally) vertex disjoint paths from $y$ to $x$ in $D_1$. Let $U = \{u_1, u_2, \ldots, u_s\}$, $W = \{w_1, w_2, \ldots, w_t\}$ and let $R = V(D_1) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices in $D_1$. If $R = \emptyset$, the path

$$P_1[u_1, x]P_2[y, w_t]$$

is a Hamiltonian path of $D_1$ through $xy$ and thus, $xy$ is also on a Hamiltonian path of $D$. So assume that $R \neq \emptyset$ and let $Q = q_1 q_2 \dots q_r$ be a Hamiltonian path of $D[R]$. If $q_r \to u_1$ or $q_r \to w_1$, the arc $xy$ is on the Hamiltonian path

$$QP_1[u_1, x]P_2[y, w_t] \quad \text{or} \quad QP_2[w_1, x]P_1[y, u_s]$$

of $D_1$ and thus, $xy$ is also on a Hamiltonian path of $D$. So $\{u_1, w_1\} \to q_r$. Let $i = \min\{j \mid N^-(q_j, U \cup W) \neq \emptyset\}$ be the minimal index such that $q_i$ has a negative neighbor in $U \cup W$ and let $j_1 = \max\{j \mid u_j \to q_i\}$ and $j_2 = \max\{j \mid w_j \to q_i\}$ be the maximal indices such that $\{u_{j_1}, w_{j_2}\} \to q_i$. If $j_1 < s$, the path

$$P_1[u_1, u_{j_1}]q_i P_1[u_{j_1+1}, u_s]$$

together with $P_2$ yields a contradiction to the choice of $P_1$ and $P_2$ and if $j_2 < t$, the path

$$P_2[w_1, w_{j_2}]q_i P_2[w_{j_2+1}, w_t]$$

together with $P_1$ yields a contradiction to the choice of $P_1$ and $P_2$. So assume that $j_1 = s$ and $j_2 = t$. It follows that

$$Q[q_1, q_{i-1}]P_1[u_1, x]P_2[y, w_t]Q[q_i, q_r]$$

is a Hamiltonian path of $D_1$ through $xy$ and thus, $xy$ is also on a Hamiltonian path of $D$. □

**Theorem 2.23.** *Let $D$ be a strong local tournament and let $xy$ be an arc of $D$ such that there exist at least two (internally) vertex disjoint paths from $y$ to $x$ in $D$. If there is a longest path $P$ from $y$ to $x$ in $D$ such that $D$ has an arc between an internal vertex of $P$ and $D - P$, then there exists a Hamiltonian path of $D$ that includes the arc $xy$.*

*Proof.* Suppose that the arc $xy$ is not traceable. Then both $x$ and $y$ are separating vertices of $D$ by Observation 2.14. By Theorem 2.22 we may assume, that $D - z$ is strong for every vertex $z \notin \{x, y\}$. Let $D_1, D_2, \dots, D_p$ be the strong decomposition of $D - x$, where $p \geq 2$. Since $xy$ is not traceable, we conclude that $y \notin V(D_1)$. Note that if $y$ belongs to $D_i$, all paths from $y$ to $x$ are subdigraphs of $D[V(D_i) \cup V(D_{i+1}) \cup \ldots \cup V(D_p) \cup \{x\}]$. Let $P$ be a longest path from $y$ to $x$ in $D$ that fulfills the assumption of this theorem.

If $2 \leq i \leq p - 1$, then $P$ contains all vertices of $D_i, D_{i+1}, \dots, D_p$. It follows that either there is an arc between a component $D_r$ and a component $D_s$, where $1 \leq r < i < s \leq p$, or $|V(D_i)| \geq 3$ and there is an arc between a component $D_r$ and $D_i - \{y\}$, where $r < i$. Both possibilities contradict the assumption that $y$ is a separating vertex of $D$.

So assume that $i = p$. Let $P_1$ and $P_2$ be two (internally) vertex disjoint paths from $y$ to $x$ such that $|V(P_1)| + |V(P_2)| \geq |V(P_1')| + |V(P_2')|$ for every pair $P_1', P_2'$ of (internally) vertex disjoint paths from $y$ to $x$. Let $P_1 = yu_1u_2 \dots u_sx$, $P_2 = yw_1w_2 \dots w_tx$ and let $R = V(D_p) - (V(P_1) \cup V(P_2))$ be the set of the remaining vertices of $D_p$. If $R = \emptyset$, the path

$$D_1D_2 \dots D_{p-1}P_1[u_1, x]P_2[y, w_t]$$

contradicts the assumption that $xy$ is not traceable. So assume that $R \neq \emptyset$ and let $A_1, A_2, \dots, A_q$ be the strong decomposition of $D[R]$, where $q \geq 1$. If there is an arc $u_sv$ from $u_s$ to $A_1$, let $Q$ be a Hamiltonian path of $D[R]$ that begins in $v$. Then

$$D_1D_2 \dots D_{p-1}P_2[w_1, x]P_2[y, u_s]Q$$

is a Hamiltonian path of $D$ through $xy$, again a contradiction. Hence $A_1 \rightarrow u_s$. Let $y = u_0$. If there is a vertex $v \in V(A_1)$ that has an in-neighbor on $P_1[y, u_s]$, let $i = \max\{j \mid u_j \rightarrow v\}$ be the maximal index with $u_i \rightarrow v$. But then

$$P_1[y, u_i]vP_1[u_{i+1}, x]$$

and $P_2$ are two (internally) vertex disjoint paths from $y$ to $x$ that contradict the choice of $P_1$ and $P_2$. Therefore $A_1 \rightarrow P_1[y, u_s]$. Analogously we can show that $A_1 \rightarrow P_2[y, w_t]$. So all in all we obtain $A_1 \rightarrow (D_p - A_1)$, a contradiction to the assumption that $D_p$ is strong. This final contradiction completes the proof of the theorem. ☐

In combining Theorems 2.22 and 2.23 we derive Theorem 2.20 as an immediate corollary.

### 2.1.2 Sufficient criteria for arc-traceability in terms of vertex degree

In this subsection we formulate a first sufficient criterion in terms of minimum degree for a strong in-tournament to be arc-traceable.

**Theorem 2.24.** *If $D$ is an in-tournament with $\delta(D) > \frac{n+1}{4}$, then $D$ is arc-traceable.*

*Proof.* We prove that the degree condition guarantees that $D$ is 2-connected. The latter together with Corollary 2.15 implies that $D$ is arc-traceable. Assume to the contrary that there exists a separating vertex $v$ of $D$. Let $D_1, D_2, \dots, D_p$ be the strong decomposition of $D - v$. Then $D_1$ contains a vertex with in-degree at most $(|V(D_1)|+1)/2$ and $D_p$ contains a vertex with out-degree at most $(|V(D_p)|+1)/2$. It follows that

$$\begin{aligned} 2\delta(D) &\leq \delta^-(D) + \delta^+(D) \\ &\leq \frac{|V(D_1)|+1}{2} + \frac{|V(D_p)|+1}{2} \\ &\leq \frac{n+1}{2} \end{aligned}$$

which finally implies that

$$\delta(D) \leq \frac{n+1}{4},$$

a contradiction. □

By the above theorem we obtain the following two results.

**Corollary 2.25.** *If $D$ is a local tournament with $\delta(D) > \frac{n+1}{4}$, then $D$ is arc-traceable.*

**Corollary 2.26** (Busch [25] 2005)**.** *If $D$ is a tournament with $\delta(D) > \frac{n+1}{4}$, then $D$ is arc-traceable.*

In Section 2.2 we will further improve the lower bound presented in Theorem 2.24.

## 2.1.3 Sufficient criteria for arc-traceability in terms of cycle lengths

In this subsection we show that the property that an arc of a strong local tournament $D$ belongs to a cycle of length more than $(|V(D)|+1)/2$ is sufficient for this arc to be traceable under an additional assumption. We begin with a preparatory result.

**Lemma 2.27.** *Let $D$ be a strong local tournament and let $xy$ be an arc of $D$. If $P$ is a longest path from $y$ to $x$ in $D$, then $P$ can be extended to a Hamiltonian cycle of $D$.*

*Proof.* Let $P$ be a longest path from $y$ to $x$ and let

$$C = u_1u_2 \ldots u_s v_0 v_1 \ldots v_r u_1,$$

where $s \geq 1$ and $r \geq 0$, be a longest cycle that includes $P$, i.e. $u_1 = y$, $u_s = x$ and $P = u_1u_2 \ldots u_s$. Now assume that $V(C) \neq V(D)$ and let $v \notin V(C)$ be a vertex

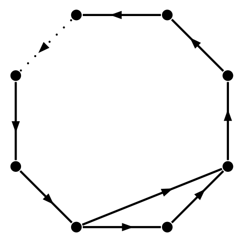

Figure 2.3: A class of local tournaments with an arc that is on a cycle of length $n-1$, but not on a Hamiltonian path.

that does not belong to $C$. If $v$ has both an out- and an in-neighbor on $C$, it can be inserted in $C$ by Lemma 1.12, a contradiction to either the choice of $P$ or $C$. It follows that the vertex set $V(D) - V(C)$ can be partitioned in $X^+ \cup X^- \cup \tilde{X}$ such that $X^- \to C \to X^+$ and the vertices of $\tilde{X}$ have neither out- nor in-neighbors on $C$. Since $D$ is strong, we conclude that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Now let $w \in X^+$ be a vertex that dominates $C$ and let $Q = w_0 w_1 \dots w_t$ be a shortest path from $C$ to $w = w_t$, where $t \geq 2$. Let $w_0^+$ and $w_0^-$ be the successor and predecessor of $w_0$ on $C$. If $w_0 \in V(P) - \{x\}$, the path

$$P[y, w_0^-]QP[w_0^+, w]$$

is a longer path from $y$ to $w$ than $P$, a contradiction. So assume that $w_0 \in V(C) - V(P)$ or $w_0 = w$. In replacing the arc $w_0 w_0^+$ by the path $Q$ we can construct a cycle that includes $P$ and is longer than $C$, the final contradiction. □

Using Lemma 2.27 for tournaments, Busch, Jacobson and Reid [26] showed the following theorem.

**Theorem 2.28** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament on $n$ vertices and let $xy$ be an arc of $T$ that is on some cycle of length more than $\frac{n+1}{2}$. Then $xy$ is on a Hamiltonian path of $T$.*

This theorem is no longer valid for local tournaments (cf. Figure 2.3).

**Example 2.29.** *Let $D$ be the local tournament on $n \geq 4$ vertices that consists of a cycle $C = v_1 v_2 \dots v_n v_1$ and the arc $v_1 v_3$. Then $v_1 v_3$ belongs to a cycle of length $n - 1 > (n+1)/2$, but not to a Hamiltonian path of $D$.*

In order to transfer Theorem 2.28 to the class of local tournaments we tighten the assumption as one can see in the next result.

**Theorem 2.30.** *Let $D$ be a strong local tournament and let $xy$ be an arc of $D$ that belongs to a longest cycle $C$ of length $\ell > \frac{n+1}{2}$. If there exists an arc between $V(C) - \{x, y\}$ and $V(D) - V(C)$, then the arc $xy$ is on a Hamiltonian path of $D$.*

*Proof.* Let $C = xyu_1u_2 \ldots u_kx$ be a longest cycle containing the arc $xy$. Using Lemma 2.27, we can extend the path $C - xy$ to a Hamiltonian cycle $C'$ of $D$. Let

$$C' = yu_1u_2 \ldots u_kxw_mw_{m-1} \ldots w_1y,$$

where $k + 2 > \frac{n+1}{2}$, $m \geq 1$ and $n = m + k + 2$. Since $C$ has length $k + 2 > \frac{n+1}{2}$, it follows that $k \geq m$. Let $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_m\}$.

*Case 1:* Suppose that there exists an arc leading from $W$ to $U$. Let

$$j = \min\{r \mid N^+(w_r, U) \neq \emptyset\}$$

be the smallest integer such that the vertex $w_j$ has at least one positive neighbor in $U$ and let

$$i = \max\{s \mid w_j \to u_i\}$$

be the greatest integer such that $w_j$ dominates $u_i$.

If $j = 1$, the path

$$w_mw_{m-1} \ldots w_1u_iu_{i+1} \ldots u_kxyu_1u_2 \ldots u_{i-1}$$

is a Hamiltonian path through $xy$.

So assume that $j \geq 2$. Note that if $u_r \to w_s \to u_{r+1}$ for two indices $1 \leq r \leq k - 1$ and $1 \leq s \leq m$, the path

$$yu_1u_2 \ldots u_rw_su_{r+1}u_{r+2} \ldots u_kx$$

is a longer path from $y$ to $x$ than $P$, a contradiction. Therefore, using the local tournament property of $D$, we conclude that $w_j \to u_r$ for every index $1 \leq r \leq i$. Since $w_j \to w_{j-1}$ and $D$ is a local tournament, it follows that $w_{j-1}$ and $u_r$ are adjacent for every index $1 \leq r \leq i$. Due to the choice of $j$ we have $u_r \to w_{j-1}$ for every index $1 \leq r \leq i$.

If $i \geq 2$, the path

$$w_mw_{m-1} \ldots w_ju_iu_{i+1} \ldots u_kxyu_1u_2 \ldots u_{i-1}w_{j-1}w_{j-2} \ldots w_1$$

is a Hamiltonian path through $xy$.

So assume that $i = 1$. Then $k \geq m \geq 2$, since $m \geq j \geq 2$. In addition note that we have already shown that $u_1 \to w_{j-1}$. Due to the choice of $P$ this implies that $u_2 \to w_{j-1}$ and, since $i = 1$ was chosen maximal, it follows that $u_2 \to w_j$. Now let

$$p = \max\{q \geq 0 \mid u_{1+q} \to w_{j-1+q}, u_{2+q} \to w_{j-1+q}, u_{2+q} \to w_{j+q}\}$$

be the greatest integer such that $u_{1+p}$ dominates $w_{j-1+p}$ and $u_{2+p}$ dominates $w_{j-1+p}$ and $w_{j+p}$.

If $p = m - j$, the path

$$u_{3+p}u_{4+p} \dots u_k x y u_1 u_2 \dots u_{2+p} w_m w_{m-1} \dots w_1$$

is a Hamiltonian path through $xy$.

So assume that $p < m - j$. Note that $2 + p < k$, since $k \geq m$. Due to the choice of $P$ we obtain $u_{3+p} \rightarrow w_{j+p}$ and thus, since $p$ was chosen maximal, we conclude that $w_{j+p+1} \rightarrow u_{3+p}$. But now

$$w_m w_{m-1} \dots w_{j+p+1} u_{3+p} u_{4+p} \dots u_k x y u_1 u_2 \dots u_{2+p} w_{j+p} w_{j-1+p} \dots w_1$$

is a Hamiltonian path through $xy$.

*Case 2:* Suppose that $U \rightsquigarrow W$. Let $u_r \rightarrow w_s$, where $1 \leq r \leq k$ and $1 \leq s \leq m$, be an arc from $U$ to $W$. Using the local tournament property of $D$ and the assumption that $U \rightsquigarrow W$, it follows that $u_r \rightarrow w_m$. But then

$$u_{r+1} u_{r+2} \dots u_k x y u_1 u_2 \dots u_r w_m w_{m-1} \dots w_1$$

is a Hamiltonian path through $xy$ which completes the proof of this theorem. □

The following results are immediate consequences of Theorem 2.30.

**Corollary 2.31** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament and let $xy \in E(T)$ be an arc of $T$ that belongs to a cycle of length $\ell > \frac{n+1}{2}$. Then the arc $xy$ is on a Hamiltonian path of $T$.*

**Corollary 2.32.** *If $D$ is a strong local tournament and every arc $xy$ of $D$ is on a longest cycle $C$ of length $\ell > \frac{n+1}{2}$ such that there exists an arc between $V(C) - \{x, y\}$ and $V(D) - V(C)$, then $D$ is arc-traceable.*

**Corollary 2.33** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a strong tournament and every arc of $T$ is on some cycle of length $\ell > \frac{n+1}{2}$, then $T$ is arc-traceable.*

## 2.2 Structure of local tournaments with a non-traceable arc

In this section we prove some important necessary conditions for local tournaments to have a non-traceable arc. These conditions are used in Section 2.3 to obtain various sufficient conditions for arc-traceability.

**Theorem 2.34.** *Let $D$ be a strong local tournament and let $xy$ be an arc of $D$ that does not belong to a Hamiltonian path. Then $D$ satisfies one of the following conditions.*

*(a)* *(i) $D$ is a round-decomposable local tournament with the round decomposition $R[H_1, H_2, \ldots, H_r]$, where $r \geq 4$, such that $V(H_i) = \{x\}$ and $V(H_j) = \{y\}$, where $|i - j| \geq 2$;*

*(ii) $D - \{x, y\}$ is not connected.*

*(b)* *(i) there exists a vertex $z$ such that $D - z$ is not strong;*

*(ii) $D - z$ has $p \geq 4$ strong components;*

*(iii) $x$ is in the initial strong component of $D - z$ and $y$ is in the terminal strong component of $D - z$;*

*(iv) $z$ has no out-neighbor in the $2^{nd}$ strong component of $D - z$ and $z$ has no in-neighbor in the $(p - 1)^{st}$ strong component of $D - z$.*

*Proof.* Let $xy$ be an arc of $D$ that is not traceable. According to Observation 2.14, both $x$ and $y$ are separating vertices of $D$.

*Case 1:* Suppose that $D$ has a separating vertex $v \notin \{x, y\}$. Then, in view of Theorem 2.22, there is at most a single vertex disjoint path leading from $y$ to $x$ in $D$. Now, by Menger's Theorem [56], there is a $y$-$x$-separating set of size one. Let $z \notin \{x, y\}$ be a separating vertex of $D$ such that there exists no path leading from $y$ to $x$ in $D - z$ and let $D_1, D_2, \ldots, D_p$ be the strong composition of $D - z$, where $p \geq 2$. According to Theorem 2.12, the vertices $y$ and $x$ are not in consecutive strong components of $D - z$ which immediately implies that $p \geq 3$. Furthermore, in view of Theorem 1.6, the vertex $z$ has at least one positive neighbor in $D_1$ and at least one negative neighbor in $D_p$. Let $x \in V(D_\alpha)$ and $y \in V(D_\beta)$.

If $\alpha > 1$, we conclude that $V(D_\alpha) = \{x\}$, since $x$ is a separating vertex of $D$. Analogously we obtain $V(D_\beta) = \{y\}$ if $\beta < p$.

If there is an arc leading from $D_i$ to $D_j$ such that $i < \alpha < j$ or $i < \beta < j$, the vertex $x$ or $y$, respectively, is not a separating vertex of $D$, a contradiction. If $\alpha > 1$ and there is an arc leading from $z$ to $D_j$ such that $\alpha < j$, the digraph $D - x$ is strong, a contradiction. If $\beta < p$ and there is an arc leading from $D_i$ to $z$ such that $i < \beta$, the digraph $D - y$ is strong, again a contradiction. So assume the contrary.

*Subcase 1.1:* Suppose that $\alpha > 1$ and $\beta < p$. Then the round decomposition $R[H_1, H_2, \ldots, H_r]$ of $D$ can be constructed as follows.

**[Construction of the round decomposition of $D$.]**

Let $H_i = D_i$ for $i = 1, 2, \ldots, p$, $H_{p+1} = \{s\}$, where $s$ is the considered separating vertex of $D$, and $r = p + 1$. Let $R$ be the local tournament with vertex set $\{H_1, H_2, \ldots, H_r\}$ and arc set $\{H_iH_j | N^+(H_i, H_j) \neq \emptyset\}$. Then $R[H_1, H_2, \ldots, H_r]$ is the desired round decomposition of $D$.

*Subcase 1.2:* Suppose that $\alpha > 1$ and $\beta = p$ or $\alpha = 1$ and $\beta < p$. Without loss of generality, we may assume the latter. By the observations above we conclude that $z \to D_1$. Since $x$ is a separating vertex of $D$, it follows that $V(D_1) = \{x\}$. In addition, if there is an arc from $z$ to $D_j$ for $j < \beta$, we obtain $z \to D_2$. In this case

$$xyD_{\beta+1}D_{\beta+2} \ldots D_p z D_2 D_3 \ldots D_{\beta-1}$$

is a Hamiltonian path of $D$ through $xy$, a contradiction. So there is no arc between $z$ and $D_j$ for $1 < j < \beta$ and the round decomposition $R[H_1, H_2, \ldots, H_r]$ of $D$ can be constructed as above.

*Subcase 1.3:* Suppose that $\alpha = 1$ and $\beta = p$. Since $x$ and $y$ are both separating vertices of $D$, we conclude that $z \not\to D_1$ if $|V(D_1)| \geq 3$ and $D_p \not\to z$ if $|V(D_p)| \geq 3$. In addition, $z$ has no positive neighbor in $D_2$ and no negative neighbor in $D_{p-1}$. If $p \geq 4$, the local tournament $D$ has the desired structure (b). So assume that $p = 3$. Then $z$ has neither an out- nor an in-neighbor in $D_2$. It follows that $D_3 \to z \to D_1$. By the observations above we conclude that $|V(D_1)| = |V(D_3)| = 1$ and the round decomposition $R[H_1, H_2, H_3, H_4]$ of $D$ can be constructed as above.

*Case 2:* Suppose that $D - v$ is strong for every vertex $v \neq x, y$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - x$, where $p \geq 2$, and let $y \in V(D_\beta)$. Since $xy$ is not traceable, we obtain $\beta > 1$.

*Subcase 2.1:* Suppose that $\beta < p$. If $|V(D_\beta)| \geq 3$, let $C_\beta$ be a Hamiltonian cycle of $D_\beta$. Then the path

$$D_1 D_2 \ldots D_{\beta-1} C_\beta[y^+, y^-] D_{\beta+1} D_{\beta+2} \ldots D_p xy$$

is a Hamiltonian path through $xy$, a contradiction. So assume that $|V(D_\beta)| = 1$. Since $y$ is a separating vertex of $D$, we conclude that $D$ has no arc leading from $D_i$ to $D_j$ for $i < \beta < j$. If $x$ has a negative neighbor in $D_i$ for an index $i < \beta - 1$, the vertex $x$ has negative neighbors both in $D_i$ and $D_p$. It follows that $D_{\beta-1} \to D_{\beta+1}$, a contradiction to the assumption that $y$ is a separating vertex of $D$. So $x \to D_i$ for every index $i \leq \beta$. We can analogously show that $x$ has no negative neighbors in $D_j$ for every index $j > \beta$. The latter particularly implies that $D_p \to x$. So the round decomposition $R[H_1, H_2, \ldots, H_r]$ of $D$ can be constructed as above.

*Subcase 2.2:* Suppose that $\beta = p$. Then $|V(D_p)| \geq 3$, since $x$ has a negative neighbor in $D_p$. Now, by Menger's Theorem [56] there are two (internally) vertex disjoint paths from $y$ to $x$. Observe that all internal vertices of these paths are in $D_p$ and that $D_{p-1} \to D_p$. Hence $xy$ is traceable by Theorem 2.23, the final contradiction. □

We make the following observations.

**Remark 2.35.** *Every strong round-decomposable local tournament that satisfies condition (a) of Theorem 2.34 is not a tournament. Furthermore, every strong tournament that satisfies condition (b) of Theorem 2.34 is not round-decomposable. In other words: if $D$ is a strong round-decomposable tournament, then $D$ is arc-traceable.*

As an immediate consequence of the above theorem and remark we obtain Theorem 2.9 as a corollary.

**Corollary 2.36** (Busch, Jacobson & Reid [26] 2006)**.** *If $T$ is a strong tournament with an arc $xy$ that is not on a Hamiltonian path, then*

*(a) there exists a vertex $z$ such that $T - z$ is not strong;*

*(b) $T - z$ has $p \geq 4$ strong components;*

*(c) $x$ is in the initial strong component of $T - z$ and $y$ is in the terminal strong component of $T - z$;*

*(d) $z$ is dominated by the $2^{nd}$ strong component of $T - z$ and $z$ dominates the $(p-1)^{st}$ strong component of $T - z$.*

A deeper analysis of the structure of strongly connected local tournaments that have a non-traceable arc and are not round-decomposable yields the following result.

**Theorem 2.37.** *Let $D$ be a strong local tournament and let $xy$ be a non-traceable arc of $D$. Let $z \notin \{x, y\}$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - z$, where $p \geq 4$. If $D$ is not round-decomposable, then $D$ has the following properties.*

*(a) If $\delta(D) \geq 2$, then the components $D_1$ and $D_p$ both have at least $2\delta(D) + 1$ vertices;*

*(b) If $D_1 - x$ induces a strong tournament in $D$, then $x$ is the only out-neighbor of $z$ in $D_1$;*

*(c) If $x$ is a separating vertex of $D_1$, then all out-neighbors of $z$ in $D_1 - x$ are in the terminal strong component of $D_1 - x$;*

*(d) If $D_p - y$ induces a strong tournament in $D$, then $y$ is the only in-neighbor of $z$ in $D_p$;*

*(e) If $y$ is a separating vertex of $D_p$, then all in-neighbors of $z$ in $D_p - y$ are in the initial strong component of $D_p - y$.*

*Proof.* Assume that $D_1 - x$ is strong. If $z$ has an out-neighbor $w$ in $D_1 - x$, let $C_p$ be a Hamiltonian cycle of $D_p$, let $v$ be an in-neighbor of $z$ in $D_p$ and let $C_1$ be a Hamiltonian cycle of $D_1 - x$. Then the path

$$xC_p[y, v]zC_1[w, w^-]D_2D_3 \dots D_{p-1}C_p[v^+, y^-]$$

shows that $xy$ is traceable, a contradiction. Hence $z$ has no out-neighbors in $D_1 - x$. So (b) is valid and by symmetry (d) is also valid.

To prove (c) assume that $x$ is a separating vertex of $D_1$. Let $A_1, A_2, \dots, A_q$ be the strong decomposition of $D_1 - x$, where $q \geq 2$. If $z$ has an out-neighbor $w$ in $A_i$, where $i < q$, let $C_p$ be a Hamiltonian cycle of $D_p$, let $v$ be an in-neighbor of $z$ in $D_p$, let $C_i$ be a Hamiltonian cycle of $A_i$ and let $Q$ be a Hamiltonian path of $D_1 - (V(A_i) \cup \{x\})$ that begins in $A_1$ and ends in an in-neighbor $u$ of $x$ in $A_q$. Then the path

$$QxC_p[y, v]zC_i[w, w^-]D_2D_3 \dots D_{p-1}C_p[v^+, y^-]$$

shows that $xy$ is traceable, again a contradiction. Hence $z$ has no out-neighbors in $A_i$ for every index $i < q$. So (c) is proved and by symmetry we derive the validity of (e).

Now let $\delta(D) \geq 2$. Then $|V(D_1)| \geq 3$. If $D_1 - x$ is strong, we deduce from (b) and Lemma 1.31.(c) that $|V(D_1) - \{x\}| \geq 2\delta(D)$ and hence $|V(D_1)| \geq 2\delta(D) + 1$. If $x$ is a separating vertex of $D_1$, we obtain $|V(A_1)| \geq 2\delta(D) - 1$ by (c) and Lemma 1.31.(b). It follows that $|V(D_1)| \geq |V(A_1)| + |V(A_2)| + |\{x\}| \geq 2\delta(D) + 1$. By symmetry we also obtain $|V(D_p)| \geq 2\delta(D) + 1$ which completes the proof of (a) and of this theorem. □

## 2.3 Applications of the structural results

In this section we shall apply the structural results of Section 2.2 to present several sufficient conditions for the arc-traceability of strongly connected local tournaments. Firstly we consider round-decomposable local tournaments. Using Lemma 1.31.(a), we can show the following result.

**Theorem 2.38.** *Let $D$ be a strong round-decomposable local tournament that is not arc-traceable and let $k$ be the number of separating vertices of $D$. If $\delta(D) \geq 2$ and $d^-(u) + d^+(v) \geq s$ for every arc $uv$ in $D$, then $ks \leq |V(D)|$.*

*Proof.* If $s = 1$, the proposition is immediate. So assume that $s \geq 2$. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of $D$, where $r \geq 4$. Note that an arbitrary Hamiltonian cycle of $D$ has no consecutive separating vertices, since $\delta(D) \geq 2$. Let $xy$ be a non-traceable arc of $D$. Since $D$ is round-decomposable, it has the structure as described in Theorem 2.34.(a). So let, without loss of generality, $V(H_1) = \{x\}$ and $V(H_i) = \{y\}$, where $3 \leq i \leq r-1$. Then $H_2$ contains a vertex $u$ and $H_{i-1}$ contains a vertex $v$ such that $d^-(u) \leq (|V(H_2)|+1)/2$, $d^+(v) \leq (|V(H_{i-1})|+1)/2$ and $u \to v$. It follows that

$$\begin{aligned} d^-(u) + d^+(v) &\leq \frac{|V(H_2)|+1}{2} + \frac{|V(H_{i-1})|+1}{2} \\ &= \frac{|V(H_2)| + |V(H_{i-1})|}{2} + 1 \\ &\leq \frac{|V(D)| - k - (k-2)(s-1)}{2} + 1. \end{aligned}$$

The last inequality is true because of Lemma 1.31.(a). Since $s \leq d^-(u) + d^+(v)$, we conclude from the last inequality that $ks \leq |V(D)|$. □

Similarly we can show the following result with the help of Lemma 1.31.(b).

**Theorem 2.39.** *Let $D$ be a strong round-decomposable local tournament that is not arc-traceable and let $k$ be the number of separating vertices of $D$. If $\delta(D) \geq 2$, then $2k\delta(D) \leq |V(D)|$.*

*Proof.* By Observation 2.14, we assume that $k \geq 2$. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of $D$, where $r \geq 4$. Note that an arbitrary Hamiltonian cycle of $D$ has no consecutive separating vertices, since $\delta(D) \geq 2$. Since $D$ is round-decomposable, it has the structure as described in Theorem 2.34.(a). So let, without loss of generality, $x$ and $y$ be two separating vertices of $D$ such that

$V(H_1) = \{x\}$ and $V(H_i) = \{y\}$, where $3 \leq i \leq r-1$. Then $H_2$ contains a vertex $u$ such that $\delta(D) \leq d^-(u) \leq (|V(H_2)| + 1)/2$. It follows that

$$\delta(D) \leq \frac{|V(H_2)| + 1}{2} \leq \frac{|V(D)| - (k-1) - (k-1)(2\delta(D) - 1)}{2}.$$

The last inequality is true because of Lemma 1.31.(b). We conclude from the last inequality that $2k\delta(D) \leq |V(D)|$. □

As immediate corollaries we state the following sufficient conditions for a strong round-decomposable local tournament to be arc-traceable.

**Corollary 2.40.** *Let $D$ be a strong round-decomposable local tournament on $n$ vertices with $k$ separating vertices. If $\delta(D) \geq 2$ and $d^-(x) + d^+(y) \geq \frac{n+1}{k}$ for every arc $xy$ of $D$, then $D$ is arc-traceable.*

*Proof.* If $D$ is not arc-traceable, it follows by Theorem 2.38 that

$$n \geq \frac{n+1}{k}k = n+1,$$

a contradiction. □

**Corollary 2.41.** *Let $D$ be a strong round-decomposable local tournament on $n$ vertices with $k$ separating vertices. If $\delta(D) \geq 2$ and $\delta(D) \geq \frac{n+1}{2k}$, then $D$ is arc-traceable.*

*Proof.* If $D$ is not arc-traceable, it follows by Theorem 2.39 that

$$n \geq \frac{n+1}{2k}2k = n+1,$$

a contradiction. □

We now consider local tournament that are not round-decomposable.

**Theorem 2.42.** *Let $D$ be a strong local tournament on $n$ vertices that is not round-decomposable with $\delta(D) \geq 2$. If $d^-(x) + d^+(y) \geq \frac{n}{2} - 2$ for every arc $xy$ of $D$, then $D$ is arc-traceable.*

*Proof.* We show that if $D$ is not traceable, it has an arc $xy$ with $d^-(x) + d^+(y) < \frac{n}{2} - 2$. So suppose that $xy$ is an arc of $D$ that is not traceable. Then $D$ has the structure as described in Theorem 2.34.(b). Note that $n \geq |V(D_1)| + |V(D_p)| + 3$ and that $D_1 \to D_p$, since $x \to y$. Let $u$ be a vertex in $D_1$ with minimal in-degree and let $v$ be a vertex in $D_p$ with minimal out-degree. Since $D_1 \rightarrow D_p$, the arc $uv$ exists in $D$. As $\delta(D) \geq 2$ we obtain $|V(D_1)| \geq 3$ and $|V(D_p)| \geq 3$.

If $D_1$ is regular or almost-regular, the vertex $x$ is not a separating vertex of $D_1$. It follows by Theorem 2.37.(b) that $(D_1 - x) \rightsquigarrow z$. Analogously, if $D_p$ is regular or almost-regular, it follows by Theorem 2.37.(d) that $z \rightsquigarrow (D_p - y)$. Hence $d^-(u) \leq (|V(D_1)| - 1)/2$ and $d^+(v) \leq (|V(D_p)| - 1)/2$.

If $D_1$ is neither regular nor almost-regular, it is immediate that $d^-(u) \leq (|V(D_1)| - 1)/2$. Analogously we see that $d^+(v) \leq (|V(D_p)| - 1)/2$ if $D_p$ is neither regular nor almost-regular.

So all in all we conclude that $d^-(u) \leq (|V(D_1)|-1)/2$ and $d^+(v) \leq (|V(D_p)|-1)/2$. It follows that

$$\begin{aligned} d^-(u) + d^+(v) &\leq \frac{|V(D_1)| - 1}{2} + \frac{|V(D_p)| - 1}{2} \\ &= \frac{|V(D_1)| + |V(D_p)| - 2}{2} \\ &\leq \frac{n-5}{2} \\ &< \frac{n}{2} - 2, \end{aligned}$$

the desired contradiction that completes the proof of this theorem. □

Since every strongly connected round-decomposable tournament is arc-traceable by Remark 2.35, the next results follow directly from Theorem 2.42.

**Corollary 2.43** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament on $n$ vertices with $\delta(T) \geq 2$. If $d^-(x) + d^+(y) \geq \frac{n}{2} - 2$ for every arc $xy$ of $T$, then $T$ is arc-traceable.*

**Corollary 2.44.** *Let $D$ be a strong local tournament on $n$ vertices that is not round-decomposable. If $\delta(D) \geq \frac{n}{4} - 1 > 1$, then $D$ is arc-traceable.*

**Corollary 2.45** (Busch, Jacobson & Reid [26] 2006)**.** *Let $T$ be a strong tournament on $n$ vertices. If $\delta(T) \geq \frac{n}{4} - 1 > 1$, then $T$ is arc-traceable.*

The next example (cf. Figure 2.4) shows that the bounds presented in Corollaries 2.40 and 2.41 are best possible.

**Example 2.46.** *Let $T_1, T_2, \ldots, T_k$ be $\sigma$-regular tournaments, where $\sigma \geq 1$, and let $u_1, u_2, \ldots, u_k$ be $k$ additional vertices. We define $D$ as the local tournament with*

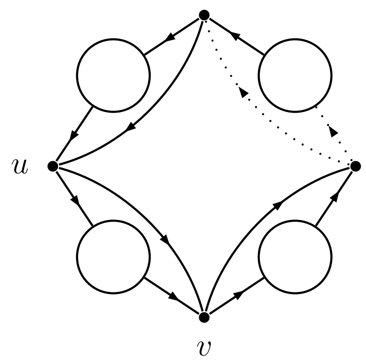


Figure 2.4: A strong round-decomposable local tournament with the non-traceable arc $uv$ which shows that Corollaries 2.40 and 2.41 are sharp.

*vertex set* $V(D) = \bigcup_{i=1}^{k} V(T_i) \cup \bigcup_{i=1}^{k}\{u_i\}$ *and arc set*

$$E(D) = \bigcup_{i=1}^{k} E(T_i) \cup \bigcup_{i=1}^{k}\{u_i v \mid v \in V(T_i)\} \cup \bigcup_{i=1}^{k}\{v u_{i+1} \mid v \in V(T_i)\} \cup \bigcup_{i=1}^{k}\{u_i u_{i+1}\}.$$

*Then $D$ is a strong round-decomposable local tournament with $\delta(D) \geq \sigma + 1$ such that $u_1, u_2, \ldots, u_k$ are the separating vertices of $D$. Note that $d^-(x) + d^+(y) = 2\sigma + 2 = \frac{n}{k}$ for all vertices $x, y \in V(T_i)$, that $d^-(u_i) + d^+(y) = d^-(x) + d^+(u_{i+1}) = 3\sigma + 3 \geq \frac{n}{k}$ for every vertex $x \in V(T_i)$ and $y \in V(T_{i+1})$ and that $d^-(u_i) + d^+(u_{i+1}) = 4\sigma + 4 \geq \frac{n}{k}$ for every index $1 \leq i \leq k$. But $D$ is not arc-traceable, since none of the arcs $u_i u_{i+1}$ belongs to a Hamiltonian path of $D$.*

Now the following question arises: if $D$ is a strong local tournament with minimal degree less than $n(D)/4$, what is the minimal length of a longest path through any given arc of $D$? The next result answers this question.

**Theorem 2.47.** *Let $D$ be a strong local tournament on $n$ vertices with minimum degree $1 < \delta(D) < \frac{n}{4}$.*

*(a) If $D$ is round-decomposable, then every arc of $D$ is on a path of length at least $2\delta(D)$;*

*(b) If $D$ is not round-decomposable, then every arc of $D$ is on a path of length at least $\lceil \frac{n+1}{2} \rceil + 2\delta(D)$.*

*Proof.* If $D$ is arc-traceable, every arc of $D$ is on a Hamiltonian path and the result is immediate. So assume that $D$ has a non-traceable arc $xy$. Then both $x$ and $y$ are separating vertices of $D$ by Observation 2.14. Furthermore, $D$ has the structure required by Theorem 2.34. We consider two cases.

*Case 1:* Suppose that $D$ is round-decomposable. Let $R[H_1, H_2, \ldots, H_r]$ be the round decomposition of $D$, where $r \geq 4$, such that, without loss of generality, $V(H_1) = \{x\}$ and $V(H_i) = \{y\}$ for an index $3 \leq i \leq r-1$. Let $D' = D[V(H_{i+1}) \cup V(H_{i+2}) \cup \ldots \cup V(H_{p-1})]$ be the local tournament that is induced by $H_{i+1}, H_{i+2}, \ldots, H_{p-1}$. Note that $N^+(D') - V(D') \subseteq \{x\}$ and $N^-(D') - V(D') \subseteq \{y\}$. It follows by Lemma 1.31.(b) that $|V(D')| \geq 2\delta(D) - 1$. Therefore the arc $xy$ is on a path with at least $2\delta(D)+1$ vertices and hence has length at least $2\delta(D)$.

*Case 2:* Suppose that $D$ is not round-decomposable. We shall show first that $z$ has an out-neighbor outside of $D_1$. If $|V(D_1)| = 1$, the proposition is immediate, since $\delta(D) \geq 2$. So assume that $|V(D_1)| \geq 3$. If $z \to D_1$, the vertex $x$ is not a separating vertex of $D$, a contradiction. Therefore $z$ has an in-neighbor $v$ in $D_1$. Since $v \to D_i$ for every index $i > 1$, it follows by the local tournament property of $D$ that $z$ is adjacent to every vertex $w \in V(D_i)$ for $i > 1$. Recall that $z$ has no in-neighbor in $D_{p-1}$. Hence we obtain $z \to D_{p-1}$. Analogously we can show that $z$ has an in-neighbor outside of $D_p$.

Let $r = \max\{i < p \mid N^-(z, D_i) \neq \emptyset\}$ be the greatest index less than $p$ such that $z$ has an in-neighbor in $D_i$. Dually, let $s = \min\{1 < j \mid N^+(z, D_j) \neq \emptyset\}$ be the smallest index greater than 1 such that $z$ has an out-neighbor in $D_j$. Observe that $r \geq 2$ and $s \leq r+1$. Let $v_1 \in V(D_r)$ and $v_2 \in V(D_p)$ be two vertices that dominate $z$ and let $w_1 \in V(D_1)$ and $w_2 \in V(D_s)$ be two vertices that are dominated by $z$. Furthermore, let $C_i$ be a Hamiltonian cycle of $D_i$ for every index $1 \leq i \leq p$. Then

$$P = C_1[x^+, x]C_p[y, v_2]zC_s[w_2, w_2^-]D_{s+1}D_{s+2}\ldots D_{p-1}C_p[v_2^+, y^-]$$

and

$$Q = C_1[x^+, w_1^-]D_2D_3\ldots D_{r-1}C_r[v_1^+, v_1]zC_1[w_1, x]C_p[y, y^-]$$

are two paths through $xy$ of order $1 + |V(D_1)| + \sum_{i=s}^{p} |V(D_i)|$ and $1 + |V(D_p)| + \sum_{i=1}^{r} |V(D_i)|$, respectively. Hence

$$\begin{aligned} |V(P)| + |V(Q)| &\geq |V(D_1)| + |V(D_p)| + 2 + \sum_{i=1}^{p} |V(D_i)| \\ &= n + 1 + |V(D_1)| + |V(D_p)| \\ &\geq n + 3 + 4\delta(D) \end{aligned}$$

by Theorem 2.37.(a). It follows that either $P$ or $Q$ contains at least $\lceil \frac{n+3}{2} \rceil + 2\delta(D)$ vertices and thus has length at least $\lceil \frac{n+1}{2} \rceil + 2\delta(D)$. □

Since every strongly connected round-decomposable tournament is arc-traceable by Remark 2.35, the following corollary is immediate by the above theorem.

**Corollary 2.48** (Busch, Jacobson & Reid [26] 2006). *Let $T$ be a strong local tournament on $n$ vertices with minimum degree $1 < \delta(T) < \frac{n}{4}$. Then every arc of $T$ is on a path of length at least $\lceil \frac{n+1}{2} \rceil + 2\delta(D)$.*

In the next result we investigate the diameter of local tournaments with a non-traceable arc.

**Theorem 2.49.** *Let $D$ be a strong local tournament with $n$ vertices and let $xy$ be an arc of $D$ that is not traceable. Then the following conditions are satisfied.*

*(a) If $D$ is round-decomposable and $C$ is a Hamiltonian cycle of $D$, then $d(y, x^+) \geq 3$ and $d(y^-, x) \geq 3$;*

*(b) If $D$ is not round-decomposable, then $d(y, v) \geq 3$ and $d(w, x) \geq 3$ for all vertices $v \in V(D_2)$ and $w \in V(D_{p-1})$.*

*Proof.* Note that $D$ has the structure required by Theorem 2.34. If $D$ is round-decomposable, it is easy to check that $d(y, x^+) \geq 3$ and $d(y^-, x) \geq 3$. Hence (a) follows. If $D$ is not round-decomposable, observe that $p \geq 4$ and that there is no arc from $z$ to $D_2$ and no arc from $D_{p-1}$ to $z$. So proposition (b) is valid. □

We call a vertex $x$ a *king* if all other vertices are in distance at most two of $x$. Then the following corollaries are immediate by Theorem 2.49.

**Corollary 2.50.** *Let $D$ be a strong local tournament on $n$ vertices. If $D$ has $n$ or $n-1$ kings, then $D$ is arc-traceable.*

**Corollary 2.51** (Busch [25] 2005). *If $T$ is a strong $n$-tournament with $n$ or $n-1$ kings, then $T$ is arc-traceable.*

## 2.4 The number of non-traceable arcs

In 2005 Busch [25] characterized the class of strongly connected tournaments whose number of non-traceable arcs is maximal to be the following.

**Definition 2.52.** *Let $T_{max}$ be the tournament that is obtained from the transitive tournament with vertex set*

$$V = \{v_0, v_1, \ldots, v_{n-1}\},$$

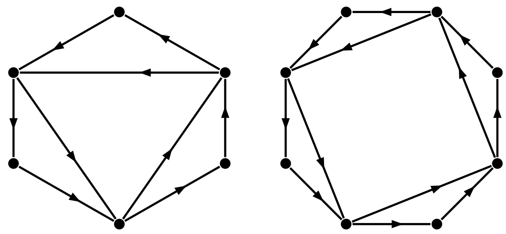

Figure 2.5: Two members of $T^{loc}_{max}$ with 6 and 8 vertices, respectively.

*where* $d^+(v_i) = i$ *for* $i = 0, 1, \ldots, n-1$, *by reversing the arcs* $v_i v_{i+2}$ *for every even index* $i < n-1$.

**Theorem 2.53** (Busch [25] 2005). *If* $T$ *is a strong tournament on* $n$ *vertices, then* $T$ *has at most* $\frac{n^2-4n+3}{8}$ *non-traceable arcs, with equality if and only if* $T$ *is isomorphic to* $T_{max}$.

The theorem above leads to the following question. If $D$ is a local tournament, how many non-traceable arcs can possibly exist in $D$? For local tournaments that fulfill condition (b) of Theorem 2.34 this question was already answered by Busch. Therefore we consider only strong round-decomposable local tournaments in the following.

**Definition 2.54.** *Let* $T^{loc}_{max}$ *be the local tournament that is defined as follows. For an even integer* $n \geq 6$ *let*

$$V = \{v_1, v_2, \ldots, v_n\}$$

*be the vertex set of* $D$. *The arc set of* $D$ *is defined as*

$$\begin{aligned} E = &\{v_n v_1\} \cup \{v_i v_{i+1} \mid i = 1, 2, \ldots, n-1\} \cup \{v_n v_2\} \\ &\cup \{v_{2j} v_{2j+2} \mid j = 1, 2, \ldots, \frac{n}{2} - 1\}. \end{aligned}$$

**Remark 2.55.** *Let* $T^{loc}_{max}$ *be a local tournament on* $n$ *vertices as defined in Definition 2.54. Then* $xy$ *is a non-traceable arc of* $T^{loc}_{max}$ *if and only if* $x = v_{2j}$ *and* $y = v_{2j+2}$ *for an index* $j \in \{1, 2, \ldots, \frac{n}{2} - 1\}$ *or* $x = v_n$ *and* $y = v_2$. *Therefore* $T^{loc}_{max}$ *has exactly* $\frac{n}{2}$ *non-traceable arcs.*

In the next theorem we characterize the class of strong round-decomposable local tournaments with maximal number of non-traceable arcs.

**Theorem 2.56.** *If* $D$ *is a strong round-decomposable local tournament, then* $D$ *has at most* $\frac{n}{2}$ *non-traceable arcs, with equality if and only if* $D$ *is isomorphic to* $T^{loc}_{max}$.

*Proof.* Recall that, according to Observation 2.14, if $uv$ is a non-traceable arc of $D$, both $u$ and $v$ are separating vertices of $D$.

Let $C$ be an arbitrary Hamiltonian cycle of $D$. Let $A = \{a_1, a_2, \ldots, a_r\}$ be the set of separating vertices of $D$ and let, without loss of generality, $a_1, a_2, \ldots, a_r$ be the order in which the vertices of $A$ appear on $C$. Assume that there exists a non-traceable arc $a_i a_j$ such that $V(C[a_{i+1}, a_{j-1}]) \cap A \neq \emptyset$. Let $a_k \in V(C[a_{i+1}, a_{j-1}]) \cap A$ be such a vertex. Then $a_i$ and $a_j$ are in the same strong component of $D - a_k$, a contradiction to Theorem 2.22. It follows that for every index $1 \leq i \leq r$ there exists at most one non-traceable arc $a_i w$. In addition, if $u$ and $v$ are consecutive vertices on $C$, then clearly there exists a Hamiltonian path through $uv$. Therefore there exist at most $\frac{n}{2}$ non-traceable arcs in $D$.

Now we characterize the equality. As we have already noted in Remark 2.55, if $D$ is isomorphic to $T^{loc}_{max}$, it has exactly $\frac{n}{2}$ non-traceable arcs. So suppose now that $D$ has exactly $\frac{n}{2}$ non-traceable arcs. We have shown above that there are no non-traceable arcs of $D$ between vertices that are consecutive on a Hamiltonian cycle of $D$ and, in addition, for each separating vertex $a_i$ of $D$ there exists at most one non-traceable arc $a_i w$. Furthermore we have seen that if $a_i a_j$ is a non-traceable arc of $D$ and $C$ is a Hamiltonian cycle of $D$, there are no separating vertices of $D$ on $C[a_{i+1}, a_{j-1}]$. Now let $C = v_1 v_2 \ldots v_n$ be a Hamiltonian cycle of $D$. We conclude that, without loss of generality, $D - v_i$ is not strong for every even index $i$ with $1 \leq i \leq n$ and that $v_{2j} v_{2j+2}$ and $v_n v_2$ are the non-traceable arcs of $D$, where $j = 1, 2, \ldots, \frac{n}{2} - 1$. It remains to show that there are no other arcs in $D$. If $xy$ is another arc of $D$, either $x = v_i$, $y = v_j$ and $|i - j| \geq 3$ or $|i - j| = 2$ and both $i$ and $j$ are odd integers. We consider the following cases.

*Case 1:* Suppose that $x = v_i$, $y = v_j$ and $|i - j| \geq 3$. In this case let $z \in V(C[v_{i+1}, v_{j-1}])$ be a separating vertex of $D$. Note that there exists at least one separating vertex $z' \in V(C[v_j, v_i])$ of $D$. The vertices on $C[v_j, v_i]$ belong to the same strong component of the strong decomposition $D - z$, a contradiction to Theorem 2.22.

*Case 2:* Suppose that $|i - j| = 2$ and both $i$ and $j$ are odd integers. Then $v_{i+1}$ is not a separating vertex of $D$, a contradiction to our assumption. The latter contradiction completes the proof. □

# Chapter 3

# Longest paths through an arc in in-tournaments

In Chapter 2 we studied the structure of local tournaments with a non-traceable arc and provided various sufficient criteria for a strongly connected local tournament to be arc-traceable. In this chapter we shall investigate 'shorter' paths through prescribed arcs in in-tournaments. The first result concerning longest paths in tournament-like digraphs is due to Rédei [60] (cf. Theorem 2.1) who showed in 1934 that every tournament has a Hamiltonian path. In 1990, Bang-Jensen [3] showed the same for connected local tournaments (cf. Corollary 1.9). Note that neither Theorem 2.1 nor Corollary 1.9 provide any information about the order of a longest path through a given arc of a (local) tournament. For tournament-like digraphs this problem has been investigated by Volkmann [73, 74, 76] and Gutin and Yeo [41]. In 2002, Volkmann [76] proved that every arc of a strongly connected tournament of order $n$ (even every arc of a strongly connected $n$-partite tournament) is contained in a path of order $\lceil (n+3)/2 \rceil$.

**Theorem 3.1** (Volkmann [76] 2002)**.** *Every arc of a strong $n$-partite tournament belongs to a path of order $\lceil \frac{n+3}{2} \rceil$.*

The following example shows that this proposition is no longer valid for strongly connected in-tournaments.

**Example 3.2** (Volkmann [74] 2000)**.** *Let $D$ be the in-tournament that consists of the cycle $x_1x_2 \dots x_nx_1$ together with the arcs $x_1x_i$ for $3 \le i \le n-1$ (cf. Figure 3.1). Then $D$ is strong, has order $n$ and the longest path through the arc $x_1x_i$ is only of order $n-i+2$.*

Let $m \in \{4, 5\}$ be an integer and let $D$ be a strongly connected in-tournament

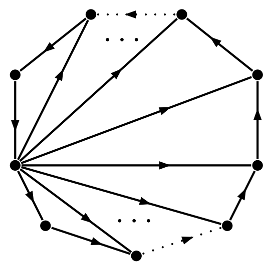


Figure 3.1: An in-tournament that shows that Theorem 3.1 is not valid for strongly connected in-tournaments.

of order $n \geq 2m - 2$ such that every arc belongs to a path of order at least $m$. In 2000, Volkmann [74] showed that if $D$ contains an arc $e$ such that the longest path through $e$ consists of exactly $m$ vertices, then $e$ is the only arc of $D$ with that property.

**Theorem 3.3** (Volkmann [74] 2000)**.** *Let $m \in \{4, 5\}$ be an integer and let $D$ be a strong in-tournament of order $n \geq 2m - 2$. If $D$ has an $m$-path arc but no $k$-path arc for $3 \leq k \leq m - 1$, then $D$ has exactly one $m$-path arc.*

In addition, Volkmann formulated the following conjecture.

**Conjecture 3.4** (Volkmann [74] 2000)**.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n \geq 2m - 2$. If $D$ has an $m$-path arc but no $k$-path arc for $3 \leq k \leq m - 1$, then $D$ has exactly one $m$-path arc.*

The next example shows that the condition $n \geq 2m - 2$ in Conjecture 3.4 is best possible.

**Example 3.5** (Volkmann [74] 2000)**.** *Let $m \geq 4$ be an integer and let the strong in-tournament $D$ consist of the cycle $C = x_1x_2 \ldots x_{m-1}y_1y_2 \ldots y_{m-2}x_1$ such that $x_1 \to C[x_2, x_{m-1}]$ and $x_{m-1} \to C[y_1, y_{m-2}]$ (cf. Figure 3.2). Then $D$ is of order $2m - 3$ and has the two $m$-path arcs $x_1x_{m-1}$ and $x_{m-1}y_{m-2}$.*

In this chapter we shall validate the conjecture of Volkmann. In addition, we will prove that if we ease the restrictions on the the order of $D$ to $n \geq 2m - 3$, the in-tournament $D$ has at most two arcs as described in Conjecture 3.4. In doing so, we will also give a characterization of the in-tournaments with exactly two such arcs.

The next observation is a first result on the order of a longest path through a given arc of an in-tournament. It was already mentioned in Chapter 2.

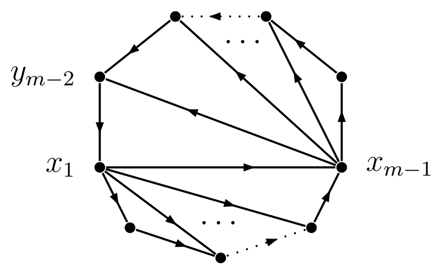


Figure 3.2: A strongly connected in-tournament on $2m-3$ vertices with two $m$-path arcs.

**Observation 2.14** (Volkmann [74] 2000). *Let $uv$ be an arbitrary arc of a strong in-tournament $D$. If $D-u$ or $D-v$ is strong, then $D$ contains a Hamiltonian path starting with the arc $uv$ or ending with the arc $uv$, respectively.*

We reformulate Corollary 2.15 in the following way.

**Corollary 3.6.** *If $uv$ is an arc of a strong in-tournament $D$ that does not belong to a Hamiltonian path, then both $u$ and $v$ are separating vertices of $D$.*

As an abbreviation we give the following notation.

**Notation 3.7.** *Let $D$ be a strong in-tournament with a separating vertex $u$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$. Furthermore, for every index $i \in \{1, 2, \ldots, p\}$ with $|V(D_i)| \geq 3$ let $C_i$ be a Hamiltonian cycle of $D_i$, for every index $1 \leq i \leq p-1$ let $w_i \in V(D_i)$ be a vertex that dominates $D_{i+1}$ and let $w_p \in V(D_p)$ be a negative neighbor of $u$. Then*

$$P = C_1[w_1^+, w_1]C_2[w_2^+, w_2] \ldots C_p[w_p^+, w_p]u$$

*is a Hamiltonian path of $D$. For $1 \leq i \leq j \leq p$ let $P_{i,j}$ denote the subpath*

$$P_{i,j} = C_i[w_i^+, w_i]C_{i+1}[w_{i+1}^+, w_{i+1}] \ldots C_j[w_j^+, w_j]$$

*of $P$ with initial vertex $w_i^+$ and terminal vertex $w_j$.*

## 3.1 Structure of in-tournaments with an $m$-path arc

Let $m \geq 4$ be an integer. In this section we investigate strongly connected in-tournaments of order $n \geq 2m-3$ that contain no $k$-path arc for $3 \leq k \leq m-1$, but at least one $m$-path arc. Example 3.5 describes a class of in-tournaments with the above properties that contain two $m$-path arcs. Let $D$ be a strongly connected in-tournament of order $n \geq 2m-3$ that contains no $k$-path arc for $3 \leq k \leq m-1$, but at least one $m$-path arc. In the following we will show that $D$ has at most two $m$-path arcs. Taking a deeper look at the structure of $D$ we shall show that if $n \geq 2m-2$, the in-tournament $D$ has only one $m$-path arc and we shall give a characterization of all strongly connected in-tournaments of order $n = 2m-3$ that contain exactly two $m$-path arcs.

We begin with a first structural result.

**Lemma 3.8.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n$ such that $D$ has no $k$-path arc for $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $uv$ is an $m$-path arc of $D$ such that $v \in V(D_p)$, then $n \leq 2m-3$.*

*Proof.* Let $uv$ be an $m$-path arc of $D$ such that $v \in V(D_p)$. Suppose that $n \geq 2m-2$. According to Corollary 1.29, the vertex $u$ has a negative neighbor $w_p$ in $D_p$ which implies that $|V(D_p)| \geq 3$. Let $vx_1x_2\ldots x_tv$ be a Hamiltonian cycle of $D_p$, where $t \geq 2$. If $t \geq m-1$, the path $uvx_1x_2\ldots x_t$ is an $(m+1)$-path through $uv$, a contradiction. So $t \leq m-2$. It follows that

$$|V(D) - (V(D_p) \cup \{u\})| = n - (t+1) - 1 \geq n - m \geq m-2$$

and thus,

$$P_{1,p-1}w_puv$$

is a path through $uv$ of order at least $m+1$, a contradiction. □

It remains to consider the case that $v$ belongs to the terminal strong component of $D-u$ and $D$ has order $n = 2m-3$.

**Lemma 3.9.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n = 2m-3$ such that $D$ has no $k$-path arc for $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_p)$ and $w = w_p$ is an in-neighbor of $u$ in $D_p$, then $uv$ is the unique $m$-path arc of $D$ and $D$ has the following structural properties.*

*(a)* $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| = m-3$;

*(b)* $|V(D_p)| = m-1$;

*(c)* $(D_p - w_p) \rightsquigarrow v \rightarrow w_p \rightsquigarrow (D_p - v)$;

*(d)* $u \rightarrow V(D) - \{u, w_p\}$.

*Proof.* Let $uv$ be an $m$-path arc of $D$ such that $v \in V(D_p)$.

(a) Since $uC_p[v, v^-]$ is a path through $uv$ and $uv$ is an $m$-path arc of $D$, it follows that $|V(D_p)| \leq m-1$. Hence $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| \geq m-3$. But if $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{p-1})| > m-3$, the path

$$P_{1,p-1}w_p uv$$

is a path through $uv$ of order at least $m+1$, a contradiction. Hence, $D$ has property (a).

(b) Property (a) immediately implies (b).

(c) Since $D_p$ is strong and $|V(D_p)| \geq 3$, it follows that $v$ has at least one positive neighbor in $D_p$ and $w_p$ has at least one negative neighbor in $D_p$. If $v$ dominates a vertex $x \neq w_p$ in $D_p$, the path

$$P_{1,p-1}w_p uvx$$

is a path of order $m+1$ through $uv$, a contradiction. So $v \to w_p$ and $(D_p - w_p) \rightsquigarrow v$. If $w_p$ is dominated by a vertex $x \neq v$ in $w_p$, the path

$$P_{1,p-1}xw_p uv$$

is an $(m+1)$-path through $uv$, a contradiction. So $w_p \rightsquigarrow (D_p - v)$ and the proof of (c) is complete.

(d) If $|N^-(u, D_p)| \geq 2$, let $x \notin \{v, w_p\}$ be a negative neighbor of $u$ in $D_p$. Then

$$P_{1,p-1}xuvw_p$$

is an $(m+1)$-path through $uv$, a contradiction. So $u$ has exactly one negative neighbor in $D_p$, the vertex $w_p$. Note that property (c) implies particularly that $w_p$ is the successor of $v$ on $C_p$. Using Theorem 1.25, it follows that $u \to (D_p - w_p)$. Note that

$$P = P_{1,p} = C_1[w_1^+, w_1]C_2[w_2^+, w_2] \ldots C_p[w_p^+, w_p]$$

is a Hamiltonian path of $D - u$ such that $u \to P[w_p^+, w_p^-]$. Using the in-tournament property of $D$, either $u \to D_1, D_2, \ldots, D_{p-1}$ or there exists a vertex $x$ on $P$ such that $x \to u \to x_P^+$. But in the latter case

$$xuC_p[v, v^-]$$

is an $(m+1)$-path through $uv$, a contradiction. This means that $D$ fulfills (d).

It remains to show that $uv$ is the unique $m$-path arc of $D$. Every arc $xy$, where $x \in V(D_i)$ and $y \in V(D_j)$ for $i+1 \leq j \leq p-1$, belongs to the path

$$xC_j[y, w_j]P_{j+1,p}$$

of order at least

$$|\{x, y\}| + \sum_{l=j+1}^{p} |V(D_l)| \geq 2 + |V(D_p)| = m + 1.$$

If $xy$ is an arc from $D_i$ to $D_p$ for an integer $i \leq p-1$, the path

$$uxC_p[y, y^-]$$

through $uv$ has order

$$|\{u, x\}| + |V(D_p)| = m + 1.$$

In the case that $xy$ is an arc of $D$ such that $x, y \in V(D_i)$, where $i \leq p-1$, we see that

$$C_p[w_p^+, w_p]uxy$$

is a path of order

$$|V(D_p)| + |\{u, x, y\}| = m + 2$$

through $xy$. Every arc $xy$ in $D_p$ belongs to the path

$$uP_{1,p-1}xyy^+$$

of order

$$|\{u, x, y, y^+\}| + \sum_{l=1}^{p-1} |V(D_l)| = m + 1.$$

If $uy$ is an arc from $u$ to $D_i$ for $1 \leq i \leq p-1$, the path

$$uC_i[y, w_i]P_{i+1,p}$$

through $uy$ has order at least

$$|\{u, y\}| + \sum_{l=i+1}^{p} |V(D_l)| \geq 2 + |V(D_p)| = m + 1.$$

In the case that $uy$ is an arc from $u$ to $D_p$ such that $y \neq v$, the path

$$P_{1,p-1}w_puyy^+$$

is a path of order

$$|\{w_p, u, y, y^+\}| + \sum_{l=1}^{p-1} |V(D_l)| = m + 1$$

through $uy$. Finally the arc $w_pu$ belongs to the Hamiltonian path $P_{1,p}u$ of $D$. Since we have discussed all possible arcs, the proof is complete. □

The next result shows that the class of in-tournaments treated in this section can be divided in two subclasses.

**Lemma 3.10.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n = 2m-3+c$, where $c \geq 0$, that contains no $k$-path arc for $3 \leq k \leq m-1$, but an $m$-path arc $uv$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_s)$ for $1 \leq s \leq p-1$, then $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$ and, in addition, $D$ has one of the following properties.*

*I.* $|V(D_s) \cup V(D_{s+1}) \cup \ldots \cup V(D_p)| = m-1$ *and*

- *(a)* $u \to D_1, D_2, \ldots, D_s$;
- *(b)* $N^-(D_i, D_j) = \emptyset$ *for every* $s+1 \leq i \leq p$ *and* $1 \leq j \leq s-1$;
- *(c)* $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_{s-1})| \geq m-3+c$.

*II.* $|V(D_s) \cup V(D_{s+1}) \cup \ldots \cup V(D_p)| = m-k$, *where* $2 \leq k \leq m-2$, *and there exists an index* $1 \leq r \leq s-1$ *and a vertex* $w_r \in V(D_r)$ *such that*

- *(a)* $u \to D_{r+1}, D_{r+2}, \ldots, D_s$;
- *(b)* $w_r \to D_{s+1}$;
- *(c)* $|V(D_1) \cup V(D_2) \cup \ldots \cup V(D_r)| = k-1$;
- *(d)* $|V(D_{r+1}) \cup V(D_{r+2}) \cup \ldots \cup V(D_{s-1})| \geq m-3+c$.

*Proof.* The path

$$P_{s+1,p}uC_s[v, v^-]$$

contains the arc $uv$ and has order $\sum_{j=s}^{p} |V(D_j)| + 1$. Since $uv$ is an $m$-path arc, it follows that $\sum_{j=s}^{p} |V(D_j)| \leq m-1$. So we consider the following two cases.

*Case I.* Suppose that $\sum_{j=s}^{p} |V(D_j)| = m-1$ which immediately implies (c).

Assume that there exists a vertex $v \neq z \in V(D_s)$ that dominates $u$ or $w_j$, where $s+1 \leq j \leq p$. Then $z \to D_{s+1}$ and hence

$$P_{1,s-1}C_s[v^+, z]P_{s+1,p}uv$$

is a path through $uv$ of order at least

$$\sum_{l=1}^{s-1} |V(D_l)| + |V(D_p)| + |\{z, u, v\}| \geq (m-3+c) + 4 \geq m+1,$$

a contradiction. So $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$ and $u \to D_s$.

Assume that there exists a vertex $z \in V(D_i)$ that dominates $u$ or $w_j$, where $1 \leq i \leq s-1$ and $s+1 \leq j \leq p$. It follows that $z \rightarrow D_{s+1}$ and thus,

$$zP_{s+1,p}uC_s[v, v^-]$$

is a path of order

$$\sum_{l=s}^{p} |V(D_l)| + |\{z, u\}| = (m-1) + 2 = m+1$$

through $uv$, a contradiction. Hence $D$ fulfills (a) and (b) which completes the proof of Case I.

*Case II.* Suppose that $\sum_{j=s}^{p} |V(D_j)| = m-k$, where $2 \leq k \leq m-2$. If $D$ has no arc leading from $\bigcup_{i=1}^{s} V(D_i)$ to $\{u\} \cup \bigcup_{j=s+1}^{p} V(D_j)$, it is easy to check that a longest path through $uv$ has order $m-1$, a contradiction. So there exists a vertex $v \neq z \in V(D_r)$, where $r \leq s$, that dominates $w_{s+1}$. If $r = s$, the path

$$P_{1,s-1}zP_{s+1,p}uv$$

through $uv$ contains at least

$$(2m-3+c) - (m-k) + 2 = m+k+c-1 \geq m+1$$

vertices, a contradiction. It follows that $N^-(D_i, D_s) \subseteq \{v\}$ for every index $s+1 \leq i \leq p$, that $u \rightarrow D_s$ and that $r \neq s$. Under the condition that $N^-(D_{s+1,D_r}) \neq \emptyset$ we choose the maximal index $r$. This choice immediately implies that $u \rightarrow D_{r+1}, D_{r+2}, \ldots, D_{s-1}$ and that, without loss of generality, $w_r \rightarrow D_{s+1}$. So (a) and (b) are true. But then

$$P_{1,r}P_{s+1,p}uC_s[v, v^-]$$

is a path through $uv$ of order $\sum_{j=1}^{r} |V(D_j)| + (m-k) + 1$. Since $uv$ is an $m$-path arc in $D$, this implies that $\sum_{j=1}^{r} |V(D_j)| \leq k-1$. Note that actually

$$\sum_{j=1}^{r} |V(D_j)| = k-1,$$

since otherwise a longest path through $uv$ contains less than $m$ vertices, a contradiction. So (c) is proved. But (c) immediately implies that

$$\sum_{j=r+1}^{s-1} |V(D_j)| \geq m-3+c$$

which shows (d) and completes the proof of Case II. □

## 3.2 Applications of the structural results

Differentiating by the properties I and II elaborated in Section 3.1, we shall now show that if a strongly connected in-tournament $D$ of order at least $2m-3$ contains an $m$-path arc $uv$, but no $k$-path arc for $3 \leq k \leq m-1$, the digraph $D$ has at most two $m$-path arcs. We start with in-tournament that belong to class I.

**Lemma 3.11.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n = 2m-3+c$, where $c \geq 0$, such that $D$ has no $k$-path arc for $3 \leq k \leq m-1$. Let $uv$ be an $m$-path arc of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D-u$, where $p \geq 2$. If $v \in V(D_s)$, where $1 \leq s \leq p-1$, and $D$ has property* I *as described in Lemma 3.10, then $D$ contains at most two $m$-path arcs.*

*Proof.* We shall show that every arc $xy \neq uv$ of $D$ is contained in a path of order at least $m+1$ with two exceptions (cf. Case 2 and Case 5).

*Case 1.* Suppose that $x \in V(D_i)$ and $y \in V(D_{i+1})$. If $i < s$, the arc $xy$ belongs to the Hamiltonian path

$$P_{1,i-1}C_i[x^+, x]C_{i+1}[y, w_{i+1}]P_{i+2,p}uC_{i+1}[w_{i+1}^+, y^-].$$

If $i \geq s$, the arc $xy$ belongs to the path

$$P_{1,i-1}C_i[x^+, x]C_{i+1}[y, w_{i+1}]P_{i+2,p}u.$$

Assume that this path has order less or equal $m$. In this case we conclude that $\sum_{j=1}^{i-1} |V(D_j)| = m-3$, $i = s$, $|V(D_i)| = |V(D_{i+1})| = 1$ and $p = s+1$. But the latter two observations imply that

$$m - 1 = \sum_{j=s}^{p} |V(D_j)| = 2 \Leftrightarrow m = 3,$$

a contradiction.

*Case 2.* Suppose that $x \in V(D_i)$ and $y \in V(D_j)$, where $i+2 \leq j$. If $i < s$, the arc $xy$ belongs to the Hamiltonian path

$$P_{1,i-1}C_i[x^+, x]C_j[y, w_j]P_{j+1,p}uP_{i+1,j-1}C_j[w_j^+, y^-].$$

If $i \geq s$, the arc $xy$ belongs to the path

$$P_{1,i-1}C_i[x^+, x]C_j[y, w_j]P_{j+1,p}u.$$

Assume that this path has order less or equal $m$. In this case we conclude that $\sum_{j=1}^{i-1} |V(D_j)| = m-3$. So $c = 0$ which implies that $n = 2m-3$. Furthermore, we observe that $V(D_s) = V(D_i) = \{x\}$ and $V(D_p) = V(D_j) = \{y\}$ which means that $x = v$ and $y = w_p$. If $D$ has an arc $uz$ such that $z \in V(D_i)$ for an index $s+1 \leq i \leq p-1$, the path

$$P_{1,s-1}xyuz$$

is a path through $xy$ and if $D$ has an arc $zu$ such that $z \in V(D_i)$ for an index $s+1 \leq i \leq p-1$, the path

$$zuP_{1,s-1}xy$$

is a path through $xy$. Both paths have order

$$\sum_{l=1}^{s-1} |V(D_l)| + |\{x, y, u, z\}| \geq (m - 3 + c) + 4 \geq m + 1.$$

So assume that no such arc exists in $D$. Then $xy = vw_p$ is an $m$-path arc in $D$.

*Case 3.* Suppose that $x, y \in V(D_i)$. If $i \geq s$, the arc $xy$ belongs to

$$uP_{1,i-1}C_i[x^-, x]y,$$

a path of order at least

$$\sum_{l=1}^{s-1} |V(D_l)| + |\{x^-, x, u, y\}| \geq (m - 3 + c) + 4 \geq m + 1.$$

If $i < s$, the arc $xy$ belongs to

$$P_{s,p}uxy,$$

a path of order

$$\sum_{l=s}^{p} |V(D_l)| + |\{u, x, y\}| = m + 2.$$

*Case 4.* Suppose that $x = u$ and $y \in V(D_j)$. Due to Lemma 3.10 we may assume that $j \leq p - 1$. If $j \geq s$ and $y \neq w_j$, the arc $xy$ belongs to

$$P_{1,j-1}C_j[y^+, w_j]P_{j+1,p}uy,$$

a path of order at least

$$\sum_{l=1}^{j-1} |V(D_l)| + |V(D_p)| + |\{w_j, u, y\}| \geq (m - 3 + c) + 4 \geq m + 1.$$

If $j < s$, the arc $xy$ belongs to

$$P_{j+1,p}uy,$$

a path of order at least

$$\sum_{l=j+1}^{p} |V(D_l)| + |\{u, y\}| \geq (m-1) + 2 = m + 1.$$

It remains to check the case that $j > s$ and $y = w_j$ is the only vertex in $D_j$ that dominates $D_{j+1}$. If $D$ has no arc leading from $\bigcup_{i=1}^{j-1} V(D_i)$ to $\{u\} \cup \bigcup_{i=j+1}^{p} V(D_i)$, a longest path through $uv$ has order $m-1$, a contradiction. So there exists a vertex $w \in V(D_i)$, where $i \leq j-1$, that dominates $D_{j+1}$. If $i < s$, note that $w \rightarrow D_{s+1}$ and thus,

$$wP_{s+1,p}uC_s[v, v^-]$$

is a path of order

$$\sum_{l=s}^{p} |V(D_l)| + |\{u, w\}| = (m-1) + 2 = m + 1$$

through $uv$, a contradiction. If $i \geq s$, the arc $uy$ belongs to

$$P_{1,i-1}C_i[w^+, w]P_{j+1,p}uC_j[y, y^-],$$

a path of order at least

$$\sum_{l=1}^{i-1} |V(D_l)| + |V(D_i)| + |V(D_p)| + |\{u, y\}| \geq (m-3+c) + 4 \geq m + 1.$$

*Case 5.* Suppose that $x \in V(D_i)$ and $y = u$. Since $u \rightarrow D_1, D_2, \ldots, D_s$, we conclude that $i \geq s+1$. But then the arc $xu$ belongs to

$$P_{1,i-1}C_i[x^+, x]u,$$

a path of order at least

$$\sum_{l=1}^{i-2} |V(D_l)| + |V(D_{i-1})| + |V(D_i)| + |\{u\}| \geq (m-3+c) + 3 \geq m.$$

Assume that this path has order less or equal $m$. In this case we conclude that $\sum_{l=1}^{i-1} |V(D_l)| = m-3$. So $c = 0$ which implies that $n = 2m-3$. Furthermore, we observe that $V(D_s) = \{v\}$ and $V(D_{s+1}) = V(D_i) = \{x\}$. If $D$ has an arc $uz$ such that $z \in V(D_i)$ for an index $i \geq s+2$, the path

$$P_{1,s-1}vxuz$$

is a path of order

$$\sum_{l=1}^{s-1} |V(D_l)| + |\{v, x, u, z\}| \geq (m-3+c) + 4 \geq m+1$$

through $xu$. So assume that no such arc exists in $D$. Then $xy = w_{s+1}u$ is an $m$-path arc in $D$.

It remains to show that the two exceptions in Case 2 and Case 5 cannot occur simultaneously. So assume that $D$ is an in-tournament with $|V(D_s)| = |V(D_{s+1})| = |V(D_p)| = 1$ and the three $m$-path arcs $uw_s$, $w_sw_p$ and $w_{s+1}u$. This is a contradiction to the structure of $D$ as elaborated in Case 5. Since we have discussed all possible arcs, the proof is complete. □

We now turn our attention to in-tournaments belonging to class II.

**Lemma 3.12.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n = 2m-3+c$, where $c \geq 0$, such that $D$ has no $k$-path arc for $3 \leq k \leq m-1$. Let $uv$ be an $m$-path arc of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $v \in V(D_s)$, where $1 \leq s \leq p-1$, and $D$ has property* II *as described in Lemma 3.10, then $uv$ is the unique $m$-path arc in $D$.*

*Proof.* We shall show that every arc $xy \neq uv$ belongs to a path of order at least $m+1$.

*Case 1.* Suppose that $xy$ is an arc of $D$ such that $x \in V(D_i)$ and $y \in V(D_{i+1})$. If $i < r$ or if $i \geq s$, the arc $xy$ belongs to

$$P_{1,i-1}C_i[x^+, x]C_{i+1}[y, w_{i+1}]P_{i+2,p}u,$$

a path of order at least

$$\begin{aligned}\sum_{l=i+2}^{p} |V(D_l)| + |\{x, y, u\}| &\geq (2m-3+c) - (k-1) - 1 + 3 \\ &= 2m - k + c \geq m+2\end{aligned}$$

and

$$\sum_{l=1}^{i-1} |V(D_l)| + |V(D_i)| + |\{u, y\}| \geq (m-3+c) + (k-1) + 3 \geq m+1,$$

respectively. If $r \leq i \leq s-1$, we can extend the path above by

$$C_{i+1}[w_{i+1}^+, y^-]$$

to a Hamiltonian path through $xy$.

*Case 2.* Suppose that $x \in V(D_i)$ and $y \in V(D_j)$ for $j \geq i+2$. If $i \geq s$, the arc $xy$ belongs to

$$P_{1,i-1}C_i[x^+, x]C_j[y, w_j]P_{j+1,p}u,$$

a path of order at least

$$\sum_{l=1}^{i-1} |V(D_l)| + |V(D_i)| + |\{u, y\}| \geq (m-3+c) + (k-1) + 3 \geq m+1.$$

If $i \leq r$ and $j \leq r$, the arc $xy$ belongs to

$$xC_j[y, w_j]P_{j+1,p}u,$$

a path of order at least

$$\sum_{l=j+1}^{p} |V(D_l)| + |\{u, x, y\}| \geq (2m-3+c) - (k-1) + 2 \geq 2m-k \geq m+2.$$

If $i \leq r$ and $j > r$, the arc $xy$ belongs to

$$xC_j[y, w_j]P_{j+1,p}uP_{r+1,j-1}C_j[w_j^+, y^-],$$

a path of order at least

$$\sum_{l=r+1}^{p} |V(D_l)| + |\{x, u\}| \geq (2m-3+c) - (k-1) + 1 \geq 2m-k-1 \geq m+1.$$

Finally, if $r+1 \leq i \leq s-1$, the path

$$P_{1,i-1}C_i[x^+, x]C_j[y, w_j]P_{j+1,p}uP_{i+1,j-1}C_j[w_j^+, y^-]$$

is a Hamiltonian path through $xy$.

*Case 3.* Suppose that $x = u$ and $y \in V(D_j)$. If $j \leq r$, the arc $uy$ belongs to

$$uC_j[y, w_j]P_{j+1,p},$$

a path of order at least

$$\sum_{l=j+1}^{p} |V(D_l)| + |\{u, y\}| \geq (2m-3+c) - (k-1) + 1 \geq 2m-k-1 \geq m+1.$$

If $r < j < s$, the arc $uy$ belongs to

$$P_{1,r}P_{s+1,p}uC_j[y, w_j]P_{j+1,s},$$

a path of order at least

$$\sum_{l=1}^{r} |V(D_l)| + \sum_{l=j+1}^{p} |V(D_l)| + |\{u, y\}| \geq (k-1) + (m-k) + 2 = m + 1.$$

If $j = s$, it follows that $y \neq v$ and thus, the arc $xy$ belongs to

$$P_{1,s-1}C_s[y^+, w_s]P_{s+1,p}uy,$$

a path of order at least

$$\sum_{l=1}^{s-1} |V(D_l)| + |\{w_s, u, y\}| \geq (m-3+c) + (k-1) + 3 \geq m + k - 1 \geq m + 1.$$

Finally, if $j > s$, either $y \neq w_j$ or $y = w_j$ and $w_j$ is the only vertex in $D_j$ that dominates $D_{j+1}$. In the first case

$$P_{1,j-1}C_j[y^+, w_j]P_{j+1,p}uy$$

is a path through $uy$ of order at least

$$\begin{aligned}\sum_{l=1}^{j-2} |V(D_l)| + |V(D_{j-1})| + |\{w_j, u, y\}| &\geq (m-3+c) + (k-1) + 4 \\ &\geq m + k \geq m + 2.\end{aligned}$$

In the the second case we observe the following. If $D$ has no arc leading from $\bigcup_{i=r+1}^{j-1} V(D_i)$ to $\{u\} \cup \bigcup_{i=j+1}^{p} V(D_i)$, a longest path through $uv$ has order $m-1$, a contradiction. So there exists a vertex $z \in V(D_i)$, where $r+1 \leq i \leq j-1$, that dominates $D_{j+1}$. It follows that $z \rightarrow D_{s+1}$ and thus,

$$P_{1,i-1}C_i[z^+, z]P_{s+1,p}uC_s[v, v^-]$$

is a path through $uv$ of order at least

$$\sum_{l=1}^{i-1} |V(D_l)| + |V(D_i)| + \sum_{l=s}^{p} |V(D_l)| + |\{u\}| \geq (k-1) + (m-k) + 2 = m + 1,$$

a contradiction.

*Case 4.* Suppose that $x, y \in V(D_i)$. If $i \geq s+1$, the arc $xy$ belongs to

$$P_{1,i-1}xy,$$

a path of order at least

$$\sum_{l=1}^{i-2} |V(D_l)| + |V(D_{i-1})| + |\{x,y\}| \geq (m-3+c) + (k-1) + 3$$
$$\geq m + k - 1 \geq m + 1.$$

If $r+1 \leq i < s$, the arc $xy$ belongs to

$$P_{1,r}P_{s,p}uP_{r+1,i-1}xy,$$

a path of order at least

$$\sum_{l=1}^{r} |V(D_l)| + \sum_{l=s}^{p} |V(D_l)| + |\{u,x,y\}| \geq (k-1) + (m-k) + 3 = m+2.$$

If $i = s$, the arc $xy$ belongs to

$$P_{1,r}P_{s+1,p}uP_{r+1,s-1}xy,$$

a path of order

$$2m - 3 + c - |V(D_s)| + 2 \geq 2m - 1 - (m-2) = m+1.$$

Finally, if $1 \leq i \leq r$, we observe the following. Firstly, any path from $y$ to any vertex $z \in V(D_i)$ that dominates $D_{i+1}$ uses the vertex $x$ and secondly, any path from $u$ to $x$ uses the vertex $y$, since otherwise it is easy to check that $xy$ belongs to a path of order at least $m+1$. It follows that every vertex of a longest path through $xy$ belongs to $V(D_1) \cup V(D_2) \cup \ldots \cup V(D_i)$. But this implies that the longest path through $xy$ is of order at most $k-1 \leq m-1$, a contradiction.

*Case 5.* Suppose that $y = u$ and $x \in V(D_i)$. It follows that either $i \leq r$ or $i > s$. In the first case the arc $xy$ belongs to

$$P_{1,i-1}C_i[x^+, x]uP_{r+1,p},$$

a path of order at least

$$\sum_{l=r+1}^{p} |V(D_l)| + |\{x,u\}| \geq (m-3+c) + (m-k) + 2 \geq m+1$$

and in the second case $xy$ belongs to

$$P_{1,i-1}xu,$$

a path of order at least

$$\sum_{l=1}^{i-2}|V(D_l)| + |V(D_{i-1})| + |\{x, u\}| \geq (m-3+c) + (k-1) + 3 \geq m+1.$$

Since we have discussed all possible arcs, the proof is complete. □

## 3.3 Conclusions and solution of a conjecture of Volkmann

Combining the lemmas of the previous section, we get the following characterization of strongly connected in-tournaments that contain an $m$-path arc.

**Theorem 3.13.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n \geq 2m-3$ such that $D$ has no $k$-path arc for $3 \leq k \leq m-1$.*

*(a) If $n \geq 2m-2$, then $D$ has at most one $m$-path arc.*

*(b) If $n = 2m-3$, then $D$ has at most two $m$-path arcs. In addition, $D$ has exactly two $m$-path arcs $uv$ and $xy$ if and only if $D$ has property* I *as described in Lemma 3.10 and $V(D_s) = \{v\}$ and either*

*(i) $v = x$, $V(D_p) = \{y\}$ and there exists no arc between $u$ and $D_j$ for every index $s+1 \leq j \leq p-1$ or*

*(ii) $y = u$, $V(D_{s+1}) = \{x\}$ and there exists no arc from $D_i$ or $u$ to $D_j$ for every index $1 \leq i \leq s$ and $s+2 \leq j \leq p$.*

*(Here $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D-u$, where $p \geq 2$.)*

As a corollary of Theorem 3.13.(a) Volkmann's Conjecture 3.4 is solved in positive.

**Corollary 3.14.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n \geq 2m-2$. If $D$ has an $m$-path arc but no $k$-path arc for $3 \leq k \leq m-1$, then $D$ has exactly one $m$-path arc.*

# Chapter 4

# Longest paths through an arc in local tournaments

Since the path structure of in-tournaments with at least $2m-2$ vertices, where $m \geq 4$, was already investigated in Chapter 3, we concentrate in this chapter on the structure of local tournaments with less than $2m-2$ vertices. In 2002, Volkmann [76] proved that every arc of a strongly connected tournament of order $n$ (even every arc of a strongly connected $n$-partite tournament) is contained in a path of order $\lceil (n+3)/2 \rceil$ (cf. Theorem 3.1). In Theorem 2.47 we showed that every arc of a strongly connected local tournament $D$ that is not round-decomposable is on a path of length at least $\lceil \frac{n+1}{2} \rceil + 2\delta(D)$ if $\delta(D) > 1$. Observe that this result is an improvement of Volkmann's result for local tournaments that are not round-decomposable and have minimal degree at least two. The following example shows that Volkmann's result is no longer valid for strongly connected local tournaments with minimal degree one (cf. Figure 4.1).

**Example 4.1.** *Let $T$ be a tournament and let $x, y, z$ be three distinct vertices that do not belong to $T$. Let $D$ be the local tournament with vertex set*

$$V = V(T) \cup \{x, y, z\}$$

*and arc set*

$$E = E(T) \cup \{xy, yz, zx\} \cup \{xv \mid v \in E(T)\} \cup \{vy \mid v \in E(T)\}.$$

*Then the longest path through $xy$ is of order three.*

In [74] Volkmann investigated longest paths through given arcs in strongly connected in-tournaments and formulated the following conjecture which was validated by Volkmann [74] for $m = 4, 5$ and Meierling for $m \geq 4$ (cf. Chapter 3).

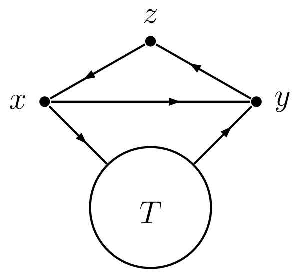


Figure 4.1: A local tournament with minimum degree $\delta = 1$ and the 3-path arc $xy$.

**Corollary 3.14.** *Let $m \geq 4$ be an integer and let $D$ be a strong in-tournament of order $n \geq 2m-2$. If $D$ has an $m$-path arc but no $k$-path arc for $3 \leq k \leq m-1$, then $D$ has exactly one $m$-path arc.*

Example 3.5 shows that the condition $n \geq 2m-2$ in Corollary 3.14 is best possible (cf. Figure 3.2). Now let $m \geq 4$ be an integer. In the following we describe the structure of strongly connected local tournaments on $n \leq 2m-3$ vertices containing an $m$-path arc but no $k$-path arc for $3 \leq k \leq m-1$. Recall that if $uv$ is an arc of a strongly connected local tournament $D$ that does not belong to a Hamiltonian path, then both $u$ and $v$ are separating vertices of $D$ (cf. Corollary 3.6).

The next result addresses some of the arcs of a strongly connected local tournaments and was already mentioned in Chapter 2.

**Theorem 2.12** (Volkmann [74] 2000)**.** *Let $u$ be a vertex of a strong local tournament such that $D-u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D-u$, then the arcs from $D_i$ to $D_{i+1}$ for $1 \leq i \leq p-1$ and the arcs in $D_i$ for $2 \leq i \leq p-1$ are contained in a Hamiltonian path of $D$.*

We would like to add the following remark.

**Remark 4.2.** *In the situation of Theorem 2.12 the arcs from $u$ to $D_1$ and the arcs from $D_p$ to $u$ are also on a Hamiltonian path of $D$.*

Observe that the structural Theorem 2.34 provides additional information about arcs that are not traceable.

## 4.1 Structure of local tournaments with an $m$-path arc

In this section let $D$ be a strongly connected local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq m$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$. By Observation 4.2 we may assume that $v$ is not in $D_1$. In the following we will consider the two cases $v \in V(D_p)$ and $v \in V(D_\alpha)$ with $2 \leq \alpha \leq p-1$.

**The case $v \in V(D_p)$.**

**Lemma 4.3.** *Let $m \geq 4$ be an integer and let $D$ be a strong local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq m$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$. If $v \in V(D_p)$, then*

*(a) $D_i \to D_j$ for every pair $i < j$ of indices;*

*(b) there is an arc between $u$ and every vertex $w \notin V(D_p)$;*

*(c) $|V(D_p)| = m - 1 - l$ and $\sum_{i=1}^{p-1} |V(D_i)| = m - s + l$, where $0 \leq l \leq s - 3$;*

*(d) every in-neighbor of $u$ in $D_p$ is a separating vertex of $D_p$;*

*(e) $v$ is a separating vertex of $D_p$;*

*(f) $u$ dominates every component of the strong decomposition of $D_p - v$ except the initial strong component.*

*Proof.* Note that, since $uv$ is not traceable, both $u$ and $v$ are separating vertices of $D$ by Corollary 3.1.

Since $u$ has positive neighbors both in $D_1$ and $D_p$, it follows gradually that $D_i \to D_j$ for every pair $i < j$ of indices. So (a) has been proved which immediately implies (b).

Let $C_p$ be a Hamiltonian cycle of $D_p$. Then $uC_p[v, v^-]$ is a path through $uv$. Since $uv$ is an $m$-path arc, it follows that

$$|V(D_p)| = m - 1 - l,$$

where $l \geq 0$ is an integer. This implies that

$$\sum_{i=1}^{p-1} |V(D_i)| = (2m - s) - (m - 1 - l) - 1 = m - s + l.$$

If $l > s - 3$, let $w$ be an in-neighbor of $u$ in $D_p$. Then $D_1 D_2 \ldots D_{p-1} wuv$ is a path through $uv$ of order at least $m + 1$, a contradiction. Hence $0 \leq l \leq s - 3$ and (c) is true.

Assume that $u$ has a negative neighbor $w$ in $D_p$ such that $D_p - w$ is strong. Then $D_1 D_2 \dots D_{p-1} wuv$ can be extended to a Hamiltonian path of $D$, again a contradiction. Therefore (d) is valid.

To prove (e) assume that $D_p - v$ is strong. It follows that $D - v$ is strong and thus, the arc $uv$ is traceable by Observation 2.14, a contradiction.

Let $A_1, A_2, \dots, A_q$ be the strong decomposition of $D_p - v$, where $q \geq 2$. Observe that $v$ has an out-neighbor in $A_1$ and an in-neighbor in $A_q$ and that $A_i \rightarrow A_j$ for every pair $i < j$ of indices. The former implies that $u$ has a neighbor in $A_q$ and the latter implies that none of the vertices in $\bigcup_{i=2}^{q-1} A_i$ is a separating vertex of $D_p$. If $u$ has an in-neighbor $z$ in $A_q$, there exists a Hamiltonian path $Q$ of $D_p - v$ that begins in $A_1$ and ends in $z$. Hence $D_1 D_2 \dots D_{p-1} Quv$ is a Hamiltonian path of $D$ through $uv$, a contradiction. So assume that $u$ has no negative neighbor in $A_q$ which implies that $u$ dominates $A_q$. Since $A_i$ dominates $A_q$ for every index $i < q$, it follows that $u$ is adjacent to all vertices of $D_p$. By (d) we conclude that $u$ dominates $A_i$ for every index $i < q$. So (f) is valid and the proof is complete. □

**The case $v \in V(D_\alpha)$, where $2 \leq \alpha \leq p - 1$.**

**Lemma 4.4.** *Let $m \geq 4$ be an integer and let $D$ be a strong local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq m$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m - 1$. Let $D_1, D_2, \dots, D_p$ be the strong decomposition of $D - u$. If $v \in V(D_\alpha)$ such that $2 \leq \alpha \leq p - 1$, then*

*(a) $D_\alpha$ consists of the single vertex $v$;*

*(b) there are no arcs leading from $D_i$ to $D_j$ for every pair $i < \alpha < j$ of indices;*

*(c) $u$ dominates $D_i$ for every index $i \leq \alpha$;*

*(d) there are no arcs leading from $D_i$ to $u$ for every index $i > \alpha$ and $u$ is dominated by $D_p$;*

*(e) $\sum_{i=1}^{\alpha-1} |V(D_i)| \geq m - s$;*

*(f) $D$ is round-decomposable and $R[D_1, D_2, \dots, D_p, u]$ is the round decomposition of $D$.*

*Proof.* Note that if $|D_i| > 1$ for an index $2 \leq i \leq p-1$, the component $D_i$ contains no separating vertex of $D$. Since $v$ is a separating vertex of $D$ by Corollary 3.1, it follows that $D_\alpha = \{v\}$ and (a) is true.

If $D$ has an arc leading from $D_i$ to $D_j$ for a pair $i, j$ of indices with $i < \alpha < j$, we conclude by Theorem 1.6 that $D_{\alpha-1} \to D_{\alpha+1}$. But then $D - v$ is strong, a contradiction. Hence (b) is valid.

To prove (c) assume that $u$ has an in-neighbor in $D_i$ for an index $i < \alpha$. Since $u$ also has an in-neighbor in $D_p$, it follows that there is an arc between $D_i$ and $D_p$, a contradiction to (b). Since $u$ dominates $v$ and $D$ is a local tournament, we conclude that $u$ dominates $D_i$ for each $i = 1, 2, \ldots, \alpha$. With the same arguments we obtain the validity of (d).

If $\sum_{i=1}^{\alpha-1} |V(D_i)| < m-s$, we obtain $\sum_{i=\alpha}^{p} |V(D_i)| \geq m$. Hence $uvD_{\alpha+1}D_{\alpha+2}\ldots D_p$ is a path of order at least $m+1$ through $uv$, a contradiction. So (e) has been proved.

Finally it is easy to check that $D$ is round-decomposable and $R[D_1, D_2, \ldots, D_p, u]$ is the round decomposition of $D$. $\square$

## 4.2 Applications of the structural results

In this section we apply the structural results of Section 4.1 to show that each strongly connected local tournament $D$ on $n$ vertices, where $\frac{3m-2}{2} \leq n \leq 2m-3$, that contains no $k$-path arc for every $k$ with $3 \leq k \leq m-1$ has at most two $m$-path arcs (cf. Theorem 4.7). As in the previous section we consider two cases.

**The case $v \in V(D_p)$.**

**Lemma 4.5.** *Let $m \geq 4$ be an integer and let $D$ be a strong local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq \frac{m+3}{2}$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m-1$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $v \in V(D_p)$, then the local tournament $D$ has at most one $m$-path arcs besides $uv$.*

*Proof.* Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - v$, where $q \geq 2$. Note that $D$ has the properties as described in Lemma 4.3. Furthermore, we observe that by Theorem 2.12 every arc $xy$ with $x \in V(D_i)$ and $y \in V(D_{i+1})$, where $1 \leq i \leq p-1$, and every arc $xy$ with $x, y \in V(D_i)$, where $2 \leq i \leq p-1$, and avery arc $uy$, where $y \in V(D_1)$, and every arc $xu$, where $x \in V(D_p)$, is traceable. In addition, note that every separating vertex $z \notin \{x, y\}$ of $D$ is in $D_1$, $A_1$ or $A_q$ and thus, every arc incident with another vertex is traceable. In distinguishing the following cases, we consider the remaining arcs of $D$.

*Case 1:* Suppose that $uy$ is an arc of $D$ such that $y \in V(A_1)$. Let $Q$ be a Hamiltonian path of $D_p - A_1 - y$ that ends in an in-neighbor of $v$ and let $w$ be an in-neighbor of $u$ in $A_1$. Then

$$P = D_1 D_2 \dots D_{p-1} wuyQv$$

is a path through $uy$. If $P$ has order less or equal $m$, we conclude that

$$|V(A_1)| \geq (2m - s) - m + 2 = m - s + 2.$$

Let $C_1$ be a Hamiltonian path of $A_1$. Then

$$D_1 D_2 \dots D_{p-1} C_1[w^+, w]uv$$

is a path through $uv$ of order greater or equal than

$$(m - s + l) + (m - s + 2) + 2 = 2m - 2s + l + 4 > m,$$

a contradiction. Hence no arc $uy$ with $y \in V(A_1)$ is an $m$-path arc of $D$.

*Case 2:* Suppose that $uy$ is an arc of $D$ such that $y \in V(A_q)$. Let $C_1$ be a Hamiltonian path of $A_1$, let $C_q$ be a Hamiltonian cycle of $A_q$ and let $w$ be an in-neighbor of $u$ in $A_1$. Then

$$P = D_1 D_2 \dots C_1[w^+, w]uC_q[y, y^-]$$

is a path through $uy$. If $P$ has order less or equal $m$, we conclude that

$$|V(A_1)| + |V(A_q)| \leq m - (m - s + l) - 1 = s - l - 1$$

and

$$\sum_{i=2}^{q-1} |V(A_i)| \geq (2m - s) - m - 1 = m - s - 1.$$

If $v$ has an out-neighbor $z \neq w$ in $A_1$, the path

$$D_1 D_2 \dots D_{p-1} wuvz A_2 A_3 \dots A_q$$

contains $uv$ and has order at least

$$(m - s + l) + (m - s - 1) + 5 = 2m - 2s + l + 4 > m,$$

a contradiction. So $A_1$ has the following structure: there exists a vertex $w$ in $A_1$ such that $(A_1 - w) \to v \to w \to u$.

If $v$ has an in-neighbor $z \neq y$ in $A_q$, the path

$$D_1 D_2 \dots D_{p-1} A_2 A_3 \dots A_{q-1} zvwuy$$

contains $uy$ and has order at least

$$(m-s+l)+(m-s-1)+5=2m-2s+l+4>m,$$

again a contradiction. So assume that $y$ is the only negative neighbor of $v$ in $A_q$. It follows that $u \to y \to v \to (A_q - y)$.

We shall show now that each arc $uz$, where $z \in V(A_q) - \{y\}$, belongs to a path of order at least $m+1$ in $D$. So let $z \neq y$ be a vertex of $A_q$ that is dominated by $u$ and let $Q$ be a Hamiltonian path of $A_1 - w$. Then $uz$ is on the path

$$D_1 D_2 \dots D_{p-1} Q A_2 \dots A_{q-1} yvwuz,$$

a path of order at least

$$(m-s+l)+(m-s-1)+5=2m-2s+l+4>m.$$

Therefore $D$ has at most one arc $uy$ with $y \in V(A_q)$ that is an $m$-path arc of $D$.

*Case 3:* Suppose that $xu$ is an arc of $D$ such that $x \in V(D_1)$. By symmetry, this case can be solved analogously to Case 1 and 2.

*Case 4:* Suppose that $xy$ is an arc of $D_1$. According to Theorem 3.1, the arc $xy$ is on a path $Q$ of order $\lceil (|V(D_1)|+3)/2 \rceil$ in $D_1$. Let $C_p$ be a Hamiltonian cycle of $D_p$ and let $w$ be an in-neighbor of $u$ in $D_p$. We observe that

$$P = Q D_2 D_3 \dots D_{p-1} C_p[w^+, w]u$$

is a path through $xy$. If $P$ has order less or equal $m$, we conclude that

$$m = 2m - s - \frac{|V(D_1)|-3}{2} \Leftrightarrow |V(D_1)| \geq 2m-2s+3 \geq m,$$

a contradiction to Lemma 4.3.(c). So no arc $xy$ of $D_1$ is an $m$-path arc of $D$.

*Case 5:* Suppose that $xy$ is an arc of $D_p$. By Theorem 3.1 the arc $xy$ is on a path $Q$ of order at least $\lceil (|V(D_p)|+3)/2 \rceil$ in $D_p$. It follows that

$$uD_1 D_2 \dots D_{p-1} Q$$

is a path through $xy$ of order greater or equal than

$$\begin{aligned} 1+(m-s+l)+\frac{(m-l-1)+3}{2} &= \frac{3m-2s+l+4}{2} \\ &\geq m+\frac{l+1}{2} > m. \end{aligned}$$

Therefore no arc $xy$ of $D_p$ is an $m$-path arc of $D$.

*Case 6:* Suppose that $xy$ is an arc of $D$ such that $x \in V(D_1)$ and $y \in V(D_p)$. Let $r < p$ be the maximal index such that $N^-(u, D_r) \neq \emptyset$ and let $s > 1$ be the minimal index such that $N^+(u, D_s) \neq \emptyset$. Furthermore, let $C_1$ be a Hamiltonian path of $D_1$, let $C_p$ be a Hamiltonian cycle of $D_p$ and let $w$ be an in-neighbor of $u$ in $D_p$. We consider two subcases.

*Subcase 6.1:* Suppose that $u$ has an out-neighbor $z \neq x$ in $D_1$ (or $u$ has an in-neighbor $z' \neq y$ in $D_p$). By symmetry we assume the former. Then

$$P = xC_p[y, w]uzD_2D_3 \ldots D_{p-1}C_p[w^+, y^-]$$

is a path through $xy$. If $P$ has order less or equal $m$, we conclude that

$$2m - s - |V(D_1)| + 2 \leq m \Leftrightarrow |V(D_1)| \geq m - s + 2.$$

Now we consider the paths

$$P_1 = C_1[x^+, x]C_p[y, w]uD_sD_{s+1} \ldots D_{p-1}C_p[w^+, y]$$

and

$$P_2 = C_1[x^+, v^-]D_2D_3 \ldots D_ruC_1[v, x]C_p[y, y^-]$$

through $xy$. Then

$$|V(P_1)| + |V(P_2)| \geq 2(|V(D_1)| + |V(D_p)| + 1) + \sum_{i=2}^{p-1} |V(D_i)|.$$

It follows that there is an index $j \in \{1, 2\}$ such that

$$\begin{aligned} |V(P_j)| &\geq |V(D_1)| + |V(D_p)| + 1 + \frac{1}{2}\sum_{i=2}^{p-1} |V(D_i)| \\ &\geq (m - s + 2) + (m - 1 - l) + 1 \\ &\geq 2m - 2s + 5 \\ &> m. \end{aligned}$$

*Subcase 6.2:* Suppose that $x$ is the only out-neighbor of $u$ in $D_1$ and $y$ is the only in-neighbor of $u$ in $D_p$. We consider the paths

$$P_1 = C_1[x^+, x^-]D_2D_3 \ldots D_ruxC_p[y, y^-]$$

and

$$P_2 = C_1[x^+, x]yuD_sD_{s+1} \ldots D_{p-1}C_p[y^+, y^-]$$

through $xy$. If $P_1$ and $P_2$ have order less or equal $m$, we conclude that

$$|V(P_1)| = \sum_{i=1}^{r} |V(D_i)| + |V(D_p)| + 1 \leq m$$

and

$$|V(P_2)| = \sum_{i=s}^{p} |V(D_i)| + |V(D_1)| + 1 \leq m.$$

It follows that

$$\sum_{i=r+1}^{p-1} |V(D_i)| \geq m - s$$

and

$$\sum_{i=2}^{s-1} |V(D_i)| \geq m - s.$$

Hence

$$|V(P_1)| + |V(P_2)| \geq 2(|V(D_1)| + |V(D_p)| + 1) + \sum_{i=2}^{p-1} |V(D_i)|$$

and thus, there is an index $j \in \{1, 2\}$ such that

$$\begin{aligned} |V(P_j)| &\geq |V(D_1)| + |V(D_p)| + 1 + \frac{1}{2}\sum_{i=2}^{p-1} |V(D_i)| \\ &\geq (m - l - 1) + 2 + (m - s) + \frac{1}{2}\sum_{i=s}^{r} |V(D_i)| \\ &\geq 2m - 2s + 4 \\ &> m. \end{aligned}$$

Therefore no arc $xy$ with $x \in V(D_1)$ and $y \in V(D_p)$ is an $m$-path arc of $D$.

So far we have shown that $D$ has at most four $m$-path arcs. We shall show now that $D$ has in fact at most two such arcs. Assume that $D$ has at least three $m$-path arcs. Then $D$ has an $m$-path arc $v'u$ such that $v' \in V(D_1)$. We consider the path

$$C_1[v'^{+}, v']uC_p[v, v^{-}]$$

through $uv$. Since $uv$ is an $m$-path arc of $D$, it follows that

$$|V(D_1)| + 1 + |V(D_p)| \leq m.$$

Therefore, either $|V(D_1)| \leq \frac{m-1}{2}$ or $|V(D_p)| \leq \frac{m-1}{2}$ and we may assume, without loss of generality, that the latter inequality holds. This implies that

$$\begin{aligned}\sum_{i=1}^{p-1} |V(D_i)| &\geq (2m-s) - 1 - \frac{m-1}{2} \\ &= \frac{3m-1}{2} - s \\ &\geq \frac{3m-1}{2} - \frac{m+3}{2} \\ &= m-2.\end{aligned}$$

But then

$$D_1 D_2 \dots D_{p-1} wuv$$

is a path of order at least $m+1$ through $uv$, a contradiction. So $D$ has at most two $m$-path arcs and the proof is complete. □

**The case $v \in V(D_\alpha)$, where $2 \leq \alpha \leq p-1$.**

**Lemma 4.6.** *Let $m \geq 4$ be an integer and let $D$ be a strong local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq \frac{m+2}{2}$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m-1$. Let $D_1, D_2, \dots, D_p$ be the strong decomposition of $D - u$. If $v \in V(D_\alpha)$, where $2 \leq \alpha \leq p-1$, the local tournament $D$ has at most one $m$-path arcs besides $uv$.*

*Proof.* Observe that $D$ has the structure as described in Lemma 4.4. Furthermore, note that, according to Theorem 3.1, all arcs $xy$ with $x \in V(D_i)$ and $y \in V(D_{i+1})$, where $1 \leq i \leq p-1$, and all arcs $xy$ with $x, y \in V(D_i)$, where $2 \leq i \leq p-1$, and all arcs $uy$, where $y \in V(D_1)$, and all arcs $xu$, where $x \in V(D_p)$, are traceable. Moreover, all separating vertices of $D$ are in a component $D_j$ with $\alpha < j < p$. Hence all arcs incident with another vertex are traceable. By Lemma 4.4.(f) the local tournament $D$ is round-decomposable and $R[D_1, D_2, \dots, D_p, u]$ is the round decomposition of $D$.

Let $xy$ be an arbitrary $m$-path arc of $D$ and let $C$ be an arbitrary Hamiltonian cycle of $D$. In view of Lemma 4.4.(e) and (f) we conclude that $\{x\} = V(D_i)$ and $\{y\} = V(D_j)$, where $|i - j| \geq 2$, and that there are at least $m - s$ vertices between $x$ and $y$ on $C$. In addition, there are no arcs in $D$ between each pair $D_r, D_s$ of components with $i < r < j$ and $j < s < i$ according to Lemma 4.4.(b). Now assume that $D$ has two $m$-path arcs besides $uv$. It follows that

$$2m - s = |V(D)| \geq 3(m-s) + 3 \Leftrightarrow s \geq \frac{m+3}{2},$$

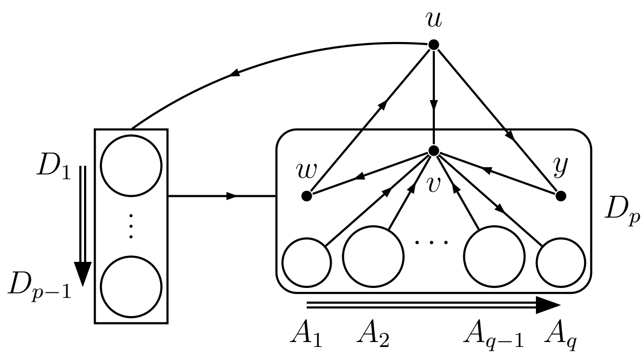


Figure 4.2: A local tournament on $2m - s$ vertices with the $m$-path arcs $uv$ and $uy$ (cf. Case $v \in V(D_p)$).

a contradiction. Therefore, the local tournament $D$ has at most one $m$-path arc besides $uv$ and the proof of this lemma is complete. □

In combining Lemmas 4.5 and 4.6 we derive the following result.

**Theorem 4.7.** *Let $m \geq 4$ be an integer and let $D$ be a strong local tournament on $n = 2m - s$ vertices, where $3 \leq s \leq \frac{m+2}{2}$, such that $uv$ is an $m$-path arc of $D$, but $D$ contains no $k$-path arc for every $k$ with $3 \leq k \leq m - 1$. Then $D$ has at most two $m$-path arcs.*

## 4.3 Concluding remarks

The structure obtained in Section 4.1 can be used to construct examples that demonstrate the sharpness of the results in Section 4.2. The next two examples show that the results presented in Lemma 4.5 and 4.6 are best possible.

**Example 4.8.** *Let $m \geq 4$ and $3 \leq s \leq \frac{m+4}{2}$ be two integers such that $s$ is even. Let $D$ be a strong local tournament on $n = 2m-s$ vertices as depicted in Figure 4.2 with*

$$\sum_{i=1}^{p-1} |V(D_i)| = m - s,$$
$$\sum_{i=2}^{q-1} |V(A_i)| = m - s,$$
$$|V(A_1)| = |V(A_q)| = \frac{s-2}{2}.$$

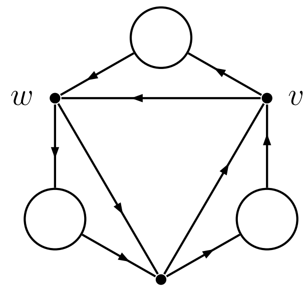


Figure 4.3: A local tournament on $2m - s$ vertices with the $m$-path arcs $uv$ and $xu$ (cf. Case $v \in V(D_\alpha)$, where $2 \leq \alpha \leq p-1$).

*If $s \leq \frac{m+3}{2}$, the arcs $uv$ and $uy$ are the only $m$-path arcs of $D$ and if $s = \frac{m+4}{2}$, the arcs $uv$, $uy$ and $wy$ are $m$-path arcs of $D$.*

**Example 4.9.** *Let $m \geq 4$ and $3 \leq s \leq \frac{m+3}{2}$ be two integers. Let $D$ be a strong local tournament on $n = 2m - s$ vertices as depicted in Figure 4.3 with*

$$\sum_{i=1}^{\alpha-1} |V(D_i)| = m - s,$$

$$\sum_{i=\alpha+1}^{\beta-1} |V(D_i)| = s - 3,$$

$$\sum_{i=\beta+1}^{p} |V(D_i)| = m - s.$$

*If $s \leq \frac{m+2}{2}$, the arcs $uv$ and $xu$ are the only $m$-path arcs of $D$ and if $s = \frac{m+3}{2}$, the arcs $uv$, $xu$ and $vx$ are $m$-path arcs of $D$.*

# Part II

# Non-separating vertices in local tournaments and in-tournaments

# Chapter 5

# Non-separating vertices in in-tournaments with high degree

From the well-known result of Moon [57] that a tournament $T$ is strongly connected if and only if it is vertex pancyclic (cf. Theorem 2.3) it follows immediately that a strongly connected tournament $T$ of order greater or equal four contains at least two non-separating vertices. This was formulated and proved by Korvin [49] in 1967.

**Corollary 5.1** (Korvin [49] 1967). *If $T$ is a strong tournament with $|V(T)| \geq 4$, then $T$ contains at least two non-separating vertices.*

In 1975, Las Vergnas [51] determined all strongly connected tournaments with exactly two non-separating vertices.

**Theorem 5.2** (Las Vergnas [51] 1975). *A strong tournament $T$ on $n$ vertices has at least three non-separating vertices, unless $T$ is isomorphic to $Q_n$, where $Q_n$ is the tournament of order $n$ consisting of a path $x_1x_2 \dots x_n$ and all arcs $x_ix_j$ for $i > j+1$.*

In 1990, Bang-Jensen [3] proved that every strongly connected locally semicomplete digraph that is not a cycle has at least one non-separating vertex.

**Theorem 5.3** (Bang-Jensen [3] 1990). *Let $D$ be a strong locally semicomplete digraph that is not a cycle. Then $D$ has a non-separating vertex.*

Four years later Guo and Volkmann showed that every strongly connected locally semicomplete digraph on $n \geq 4$ vertices has at least two non-separating vertices if it has at least $n+2$ arcs and determined the digraph in Figure 5.1 to be the only locally semicomplete digraph with exactly one non-separating vertex.

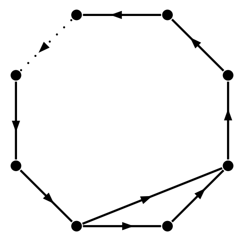

Figure 5.1: The locally semicomplete digraph with exactly one non-separating vertex.

**Theorem 5.4** (Guo & Volkmann [36] 1994)**.** *Let $D$ be a strong locally semicomplete digraph.*

*(a) If $D$ has at least $|V(D)| + 2$ arcs, then $D$ has at least two non-separating vertices;*

*(b) The digraph $D$ has exactly one non-separating vertex if and only if $D$ is isomorphic to the digraph depicted in Figure 5.1;*

*(c) Every vertex of $D$ is a separating vertex if and only if $D$ is a cycle.*

Concerning in-tournaments little is known about the number of non-separating vertices. In their initial article [16] Bang-Jensen, Huang and Prisner proved the existence of a non-separating vertex if the in-tournament in question is strongly connected and not a cycle. It is the analog of Theorem 5.3 for in-tournaments.

**Theorem 5.5** (Bang-Jensen, Huang and Prisner [16] 1993)**.** *Let $D$ be a strong in-tournament that is not a cycle. Then $D$ has a non-separating vertex.*

Tewes [65] gave a sufficient criterion for strongly connected in-tournaments to have at least two non-separating vertices.

**Theorem 5.6** (Tewes [65] 1999)**.** *Let $D$ be a strong in-tournament of order $n \geq 4$ with $\max\{\delta^+(D), \delta^-(D)\} \geq 2$. Then $D$ contains distinct vertices $x_1, x_2$ such that $D - x_i$ is strong for $i = 1, 2$.*

In 2006, Kotani [50] proved the following variants of Corollary 5.1 for strongly connected tournaments with minimum degree at least two.

**Theorem 5.7** (Kotani [50] 2006)**.** *Let $T$ be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$, then $T$ has at least $k = \min\{|V(T)|, 4p - 2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Theorem 5.8** (Kotani [50] 2006)**.** *Let $T$ be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p$, then $T$ has at least $k = 4p - 1$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Theorem 5.9** (Kotani [50] 2006)**.** *Let $T$ be a strong tournament and let $p \geq 3$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p + 1$, then $T$ has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

In this chapter we will show that Theorem 5.7 is true for the class of in-tournaments and characterize the class of in-tournaments that do not fulfill Theorems 5.8 and 5.9, thereby generalizing Kotani's results.

The next lemmas will be frequently used in our proof.

**Lemma 5.10.** *Let $D$ be a strong digraph of order $n$ and let $q$ be an integer such that $1 \leq q \leq n - 1$. Suppose that for every subset $X \subseteq V(D)$ with $1 \leq |X| \leq q$ there exists a cycle $C$ of order $n-1$ in $D$ such that $X \subseteq V(C)$. Then $D$ contains at least $q+1$ vertices $x_1, x_2, \ldots, x_{q+1}$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, q+1$.*

*Proof.* Let $X = \{x \in V(D) \mid D - x \text{ is strong}\}$ be the set of non-separating vertices of $D$. Suppose to the contrary that $|X| \leq q$. By the assumption of this lemma, there exists an $(n-1)$-cycle $C$ in $D$ such that $X \subseteq V(C)$. Let $\{v\} = V(D) - V(C)$. On the one hand this implies that $v \notin X$, but on the other hand $C$ is a Hamiltonian cycle of $D - v$ and thus, $D - v$ is strong. It follows that $v \in X$, a contradiction. □

**Lemma 5.11.** *Let $p \geq 3$ and let $D$ be an in-tournament on $2p + 1$ vertices with vertex set $W \cup \{u\}$, where $W = \{w_1, w_2, \ldots, w_{2p}\}$, such that*

$$p - 1 \leq \delta(D[W]), \Delta(D[W]) \leq p.$$

*If $d^+(w, D[W]) = p - 1$ implies that $w \to u$ for every vertex $w \in W$, then $D[W]$ is a 2-connected tournament.*

*Proof.* Firstly we will show that $D[W]$ is a tournament. Suppose, without loss of generality, that $w_1$ and $w_2$ are not adjacent. It follows that $d^+(w_1, D[W]) = d^+(w_2, D[W]) = p - 1$ and thus, $\{w_1, w_2\} \to u$ by our assumption. Using the in-tournament property of $D$, we conclude that $w_1$ and $w_2$ are adjacent, a contradiction. So $D[W]$ is a tournament.

Secondly we will show that $D[W]$ is 2-connected. Suppose, without loss of generality, that $w_1$ is a separating vertex of $D[W]$ and let $D_1, D_2, \ldots, D_q$ be the strong decomposition of $D[W] - w_1$, where $q \geq 2$. Then there exists a vertex $w \in V(D_1)$ and a vertex $w' \in V(D_q)$ such that

$$d^+(w, D[W]) \geq \frac{|V(D_1)| - 1}{2} + |V(D_2)| + \ldots + |V(D_q)|$$

and

$$d^+(w', D[W]) \leq \frac{|V(D_p)| - 1}{2} + 1.$$

Since $p - 1 \leq \delta(D[W]), \Delta(D[W]) \leq p$, it follows that $q \leq 3$ and $|V(D_i)| = 1$ for $1 \leq i \leq q$, a contradiction to the assumption that $p \geq 3$. So $D[W] - w$ is strong for every vertex $w \in W$ which means that $D[W]$ is 2-connected. □

## 5.1 The exceptional cases

In this section we introduce the classes of local tournaments and in-tournaments that satisfy the preconditions of Theorem 5.8 and Theorem 5.9, but not the conclusions.

**Definition 5.12.** *Let $p \geq 2$ be an integer and let $T_1$ and $T_2$ be two $(p-1)$-regular tournaments. Furthermore, let $u, v$ be two vertices such that $u, v \notin V(T_i)$ for $i = 1, 2$.*

*We define $\mathcal{T}^*_{loc}$ as the set of all local tournaments $D$ with vertex set*

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

*and arc set*

$$\begin{aligned} E(D) = &E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{wv \mid w \in V(T_1)\} \cup \\ &\{vw \mid w \in V(T_2)\} \cup \{wu \mid w \in V(T_2)\} \cup A, \end{aligned}$$

*where $A \in \{\{uv\}, \{vu\}, \emptyset\}$.*

*Let $p \geq 3$ be an integer. Let $T_1$ be defined as above and let $T_0$ be a tournament on $2p$ vertices such that $p - 1 \leq \delta(T_0), \Delta(T_0) \leq p$. Moreover, let $u, v$ be two vertices such that $u, v \notin V(T_i)$ for $i = 0, 1$. We define $\mathcal{T}^{**}_{loc}$ as the set of all local tournaments $D$ with vertex set*

$$V(D) = V(T_0) \cup V(T_1) \cup \{u, v\}$$

*and arc set*

$$\begin{aligned} E(D) = &E(T_0) \cup E(T_1) \cup \{uw \mid w \in V(T_1)\} \cup \{wv \mid w \in V(T_1)\} \cup \\ &\{vw \mid w \in V(T_0)\} \cup \{wu \mid w \in V(T_0)\} \cup A, \end{aligned}$$

*where $A \in \{\{uv\}, \{vu\}, \emptyset\}$.*

**Definition 5.13.** *Let $p \geq 3$ be an integer, let $T_1$ be a $(p-1)$-regular tournament and let $T_2$ be a tournament on $2p$ vertices such that $p-1 \leq \delta(T_2), \Delta(T_2) \leq p$. Furthermore, let $u, v$ be two vertices such that $u, v \notin V(T_i)$ for $i = 1, 2$.*

*We define $\mathcal{T}_{in}^{*}$ as the set of all in-tournaments $D$ with vertex set*

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

*and arc set*

$$\begin{aligned} E(D) = & E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{vw \mid w \in V(T_2)\} \cup \\ & \{wv \mid w \in V(T_1)\} \cup \{wu \mid w \in V(T_2) \text{ and } d^{+}(w, T_2) = p-1\} \cup \\ & A \cup B_1, \end{aligned}$$

*where*

$$\begin{aligned} A &\in \{\{uv\}, \{vu\}, \emptyset\} \text{ and} \\ B_1 &\subseteq \{wu \mid w \in V(T_2) \text{ and } d^{+}(w, T_2) = p\}. \end{aligned}$$

*In addition, we define $\mathcal{T}_{in}^{**}$ as the set of all in-tournaments $D$ with vertex set*

$$V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$$

*and arc set*

$$\begin{aligned} E(D) = & E(T_1) \cup E(T_2) \cup \{uw \mid w \in V(T_1)\} \cup \{vw \mid w \in V(T_2)\} \cup \\ & \{wv \mid w \in V(T_1)\} \cup \{wu \mid w \in V(T_2) \text{ and } d^{+}(w, T_2) = p-1\} \cup \\ & A \cup B_1 \cup B_2, \end{aligned}$$

*where*

$$\begin{aligned} A &\in \{\{uv\}, \{vu\}\}, \\ B_1 &\subseteq \{wu \mid w \in V(T_2) \text{ and } d^{+}(w, T_2) = p\} \text{ and} \\ \emptyset \neq B_2 &\subseteq \{uw \mid w \in V(T_2) \text{ and } d^{+}(w, T_2) = p\}. \end{aligned}$$

It is easy to check that the classes defined in the examples above do not fulfill Theorems 5.8 and 5.9, respectively.

**Remark 5.14.** *Let $D \in \mathcal{T}_{loc}^{*} \cup \mathcal{T}_{loc}^{**} \cup \mathcal{T}_{in}^{*} \cup \mathcal{T}_{in}^{**}$ be a digraph. Then $u$ and $v$ are the only separating vertices of $D$. In addition the following holds.*

*(a) If $D \in \mathcal{T}_{loc}^{*}$, then $|V(D)| = 4p$ and $\delta(D) = p$. Hence $D$ does not fulfill Theorem 5.8.*

*(b) If $D \in \mathcal{T}_{loc}^{**} \cup \mathcal{T}_{in}^{*} \cup \mathcal{T}_{in}^{**}$, then $|V(D)| = 4p+1$ and $\delta(D) = p$. Hence $D$ does not fulfill Theorem 5.9.*

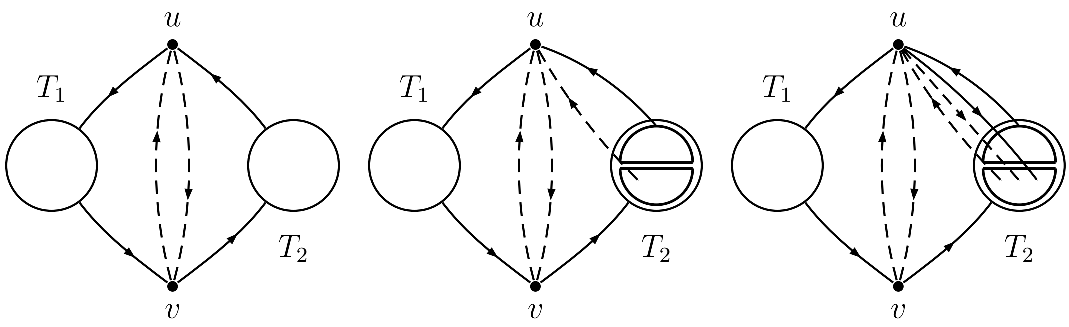


Figure 5.2: The classes $\mathcal{T}^*_{loc}$ / $\mathcal{T}^{**}_{loc}$, $\mathcal{T}^*_{in}$ and $\mathcal{T}^{**}_{in}$.

## 5.2 Structure of in-tournaments with high minimum degree

In this section we investigate the structure of in-tournaments that fulfill the assumptions of Theorems 5.8 and 5.9. Throughout this section, let $p \geq 2$ be an integer and let $D$ be a strongly connected in-tournament such that $\delta(D) \geq p$. Let $X$ be a subset of $V(D)$ such that $\emptyset \neq X \neq V(D)$. Obviously $D$ contains a strongly connected induced subdigraph $D'$ such that $V(D) - V(D') - X \neq \emptyset$ (any vertex $x \in X$ satisfies this condition). Assume now that we have chosen a strongly connected subdigraph $D'$ of $D$ under the following conditions:

**A**. $V(D) - V(D') - X \neq \emptyset$,

**B**. under condition A: $|V(D') \cap X|$ is maximal and

**C**. under condition B: $|V(D')|$ is maximal.

Note that, since $D'$ is a strongly connected in-tournament the digraph $D'$ has a Hamiltonian cycle $C = v_1v_2 \ldots v_kv_1$ according to Theorem 1.26 (if $k = 1$, then, for the sake of simplicity, the single vertex $v_1$ will in the following be called a Hamiltonian cycle). Let

$$Y = V(D) - V(C) - X$$

and define

$$\begin{aligned}
X^+ &= \left\{v \in X - V(C) \mid C \rightsquigarrow v \text{ and } N^-(v,C) \neq \emptyset\right\}, \\
X^- &= \{v \in X - V(C) \mid v \to C\}, \\
\hat{X} &= X - V(C) - X^+ - X^-, \\
Y^+ &= \left\{v \in Y - V(C) \mid C \rightsquigarrow v \text{ and } N^-(v,C) \neq \emptyset\right\}, \\
Y^- &= \{v \in Y - V(C) \mid v \to C\} \text{ and} \\
\hat{Y} &= Y - Y^+ - Y^-.
\end{aligned}$$

Note that, according to these definitions,

$$N^+(X^+,C) = N^-(X^-,C) = N^+(Y^+,C) = N^-(Y^-,C) = \emptyset. \tag{5.1}$$

Using this notation, we prove the following claims.

**Claim 1.** $\hat{X} = \{x \in X - V(C) \mid N^+(x,C) = N^-(x,C) = \emptyset\}$.

*Proof.* Suppose that there exists a vertex $x \in \hat{X}$ such that $N^+(x,C) \neq \emptyset$. Then, according to Lemma 1.25, either $x \to C$ or there exists a cycle $C'$ in $D$ such that $V(C') = V(C) \cup \{x\}$. But the first possibility is a contradiction to the definition of $\hat{X}$ and the latter possibility contradicts condition B of the choice of $C$. So $N^+(\hat{X},C) = \emptyset$.

Suppose that there exists a vertex $x \in \hat{X}$ such that $N^-(x,C) \neq \emptyset$. Note that $N^+(x,C) = \emptyset$ by the observations above. It follows that $x \in X^+$, a contradiction. So $N^-(\hat{X},C) = \emptyset$. □

Analogously to Claim 1, we can prove the following claim.

**Claim 2.** *If* $|Y| \geq 2$, *then* $N^+(\hat{Y},C) = N^-(\hat{Y},C) = \emptyset$.

**Claim 3.** $N^+(\hat{X},X^+) = N^+(\hat{X},Y^+) = \emptyset$.

*Proof.* Suppose that there exists an arc $xv$ such that $x \in \hat{X}$ and $v \in X^+ \cup Y^+$. Then $v$ has negative neighbors both in $C$ and $\hat{X}$. Since $D$ is an in-tournament, this is a contradiction to Claim 1. It follows that $N^+(\hat{X},X^+) = N^+(\hat{X},Y^+) = \emptyset$. □

Using Claim 2, we can prove the following claim analogously to Claim 3.

**Claim 4.** *If* $|Y| \geq 2$, *then* $N^+(\hat{Y},X^+) = N^+(\hat{Y},Y^+) = \emptyset$.

**Claim 5.** $N^+(X^+,X^-) = \emptyset$.

*Proof.* Suppose that there exists an arc $x_1x_2$ such that $x_1 \in X^+$ and $x_2 \in X^-$. Since $x_1 \in X^+$, we may assume, without loss of generality, that $v_k \to x_1$. But then the cycle

$$C[v_1, v_k]x_1x_2v_1$$

yields a contradiction to condition B of the choice of $C$. So $N^+(X^+, X^-) = \emptyset$. □

**Claim 6.** *Let $u, w \in V(D)$ be two vertices of $D$ such that $w \to C$ and $N^-(u, C) \neq \emptyset$. If $P$ is a path with initial vertex $u$ and terminal vertex $w$, then $Y \subseteq V(P)$.*

*Proof.* We may assume, without loss of generality, that $v_k \to u$. Suppose that $Y \not\subseteq V(P)$. Then the cycle

$$v_kPC[v_1, v_k]$$

yields a contradiction to condition C of the choice of $C$. So $Y \subseteq V(P)$. □

**Claim 7.** $|Y^+|, |Y^-| \leq 1$.

*Proof.* Suppose that $|Y^+| \geq 2$. Let $P = z_0z_1 \ldots z_r$ be a shortest path in $D$ such that $z_0 \in Y^+$ and $N^+(z_r, C) \neq \emptyset$. Then $V(P) \cap Y^+ = \{z_0\}$ and $z_r \in X^- \cup Y^-$, but $V(P) \not\subseteq Y$, a contradiction to Claim 6. □

**Claim 8.** *If $|Y^+| = |Y^-| = 1$, then $N^+(X^+, Y^-) = N^+(Y^+, X^-) = \emptyset$.*

*Proof.* Suppose that there exists an arc $uw$ such that $u \in X^+$ and $w \in Y^-$ or $u \in Y^+$ and $w \in X^-$. We may assume, without loss of generality, that $v_k \to u$. Then

$$C[v_1, v_k]uwv_1$$

contradicts condition B of the choice of $C$. So $N^+(X^+, Y^-) = N^+(Y^+, X^-) = \emptyset$. □

**Claim 9.** *Let $D$ be a strong in-tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as above. Let $X - V(C) \neq \emptyset$. If $|Y| \geq 2$, then $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$.*

*Proof.* Assume to the contrary that $X^- \cup Y^- = \emptyset$. Note that, according to Claim 7, $|Y^+|, |Y^-| \leq 1$ and, according to Claim 1, $N^+(\hat{X}, C) = \emptyset$. Since $|Y| \geq 2$ and $Y^- = \emptyset$, we conclude that $\hat{Y} \neq \emptyset$. Since $D$ is strong, it follows that $N^+(\hat{Y}, C) \neq \emptyset$, a contradiction to Claim 2. So $X^- \cup Y^- \neq \emptyset$. We can analogously show that $X^+ \cup Y^+ \neq \emptyset$. □

## 5.3 Applications of the structural results

In this section we characterize $\mathcal{T}^*_{loc}$ and $\mathcal{T}^{**}_{loc}$ as the classes of local tournaments that are exceptions for Theorem 5.8 and Theorem 5.9, respectively. Furthermore we characterize $\mathcal{T}^*_{in}$ and $\mathcal{T}^{**}_{in}$ as the classes of in-tournaments that are exceptions for Theorem 5.9. Using the preparatory structural results of Section 5.2, we show the following lemmas.

**Lemma 5.15.** *Let $D$ be a strong in-tournament and let $C$ ,$X^+$, $X^-$, $\hat{X}$, $X$, $Y^+$, $Y^-$, $\hat{Y}$, $Y$ be defined as above. Let $\delta(D) \geq p \geq 2$ and let $X - V(C) \neq \emptyset$. If $|X| = |V(D)| - 1$, then $|X| \geq 4p - 2$.*

*Proof.* Note that $V(C) \subseteq X$ by condition A of the choice of $C$. Let $V(D) - X = Y = \{y\}$.

*Case 1:* Suppose that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. We define the sets

$$A = \left\{v \in V(D) \mid \text{there exists a path leading from } X^+ \text{ to } v \text{ in } D - y\right\}$$

and

$$B = \left\{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } X^- \text{ in } D - y\right\}.$$

Note that $A \subseteq X^+ \cup \hat{X}$ and $B \subseteq X^- \cup \hat{X}$. Furthermore, $A \cap B = \emptyset$ by Claim 6. In addition, $N^+(A) - A \subseteq \{y\}$ and $N^-(B) - B \subseteq \{y\}$. Using Lemma 1.31.(b), it follows that $|A|, |B| \geq 2p - 1$ and thus,

$$|X| \geq |A| + |B| \geq 4p - 2.$$

*Case 2:* Suppose that $X^+ = \emptyset$. In order to show that $|V(C)| \geq 2p-1$ we consider the three subcases $Y = \hat{Y}$, $Y = Y^+$ and $Y = Y^-$.

*Subcase 2.1:* Suppose that $Y = \hat{Y}$. By (5.1) and Claim 1, it follows that $N^+(C) - V(C) \subseteq \{y\}$. Using Lemma 1.31.(b), we conclude that $|V(C)| \geq 2p - 1$.

*Subcase 2.2:* Suppose that $Y = Y^+$. By Claim 1, it follows that $N^+(C, \hat{X}) = \emptyset$ and thus, $N^+(C) - V(C) \subseteq \{y\}$. Using Lemma 1.31.(b), we conclude that $|V(C)| \geq 2p - 1$.

*Subcase 2.3:* Suppose that $Y = Y^-$. We derive $N^+(C) \subseteq V(C)$, a contradiction to the assumption that $D$ is strong.

Note that $N^-(X^-, C) = \emptyset$ by definition and $N^-(\hat{X}, C) = \emptyset$ by Claim 1. Hence $N^-(X^- \cup \hat{X}) - (X^- \cup \hat{X}) \subseteq \{y\}$. Using Lemma 1.31.(b), it follows that $|X^- \cup \hat{X}| \geq 2p - 1$ and thus,

$$|X| \geq |V(C)| + |X^+ \cup \hat{X}| \geq 4p - 2.$$

The case $X^- = \emptyset$ can be solved analogously to Case 2 which completes the proof of this lemma. □

**Lemma 5.16.** *Let $D$ be a strong local tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as above. Let $X - V(C) \neq \emptyset$ and let $|X| \leq |V(D)| - 2$.*

*(a) If $\delta(D) \geq p \geq 2$, then $|X| \geq 4p - 1$ or $|X| = 4p - 2$ and $D \in \mathcal{T}^*_{loc}$.*

*(b) If $\delta(D) \geq p \geq 3$, then $|X| \geq 4p$ or $|X| = 4p - 1$ and $D \in \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ or $|X| = 4p - 2$ and $D \in \mathcal{T}^*_{loc}$.*

*Proof.* Recall that $|Y| \geq 1$. If $|Y| = 1$, we conclude $|V(C) - X| \geq 1$, since $|X| \leq |V(D)| - 2$. If $|Y| \geq 2$, we derive by Claim 9 that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$. We will now consider the cases $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$ (Case 1), $X^+ \cup Y^+ = \emptyset$ (Case 2) and $X^- \cup Y^- = \emptyset$ (Case 3).

*Case 1:* Suppose that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$.

*Subcase 1.1:* Suppose that $\hat{Y} = \emptyset$. Then $Y^+ \cup Y^- \neq \emptyset$.

*Subcase 1.1.1:* Suppose that $X^+ \neq \emptyset$ and $Y^+ \neq \emptyset$. Let $Y^+ = \{y\}$. We define the sets

$$A = \{v \in V(D) \mid \text{there exists a path leading from } X^+ \text{ to } v \text{ in } D - y\}$$

and

$$B = \{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } X^- \cup Y^- \text{ in } D - y\}.$$

Note that $A \subseteq X^+ \cup \hat{X}$ and $B - Y^- \subseteq X^- \cup \hat{X}$. Furthermore, $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an out-branching of $D[A]$, where $q \geq 1$, and let $B_1, B_2, \ldots, B_r$ be an out-branching of $D[B]$, where $r \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{y\}$ and thus, $|V(A_q)| \geq 2p - 1$ by Lemma 1.31.(b). Analogously, $N^-(B_1) - V(B_1) \subseteq \{y\}$ and thus, $|V(B_1)| \geq 2p - 1$ by Lemma 1.31.(b). Let $C_q$ be a Hamiltonian cycle of $A_q$. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X| \geq 3$ by choice of $C$. It follows that

$$|X| \geq |V(C) \cap X| + |V(A_q)| + |V(B_1) - Y^-| \geq 4p.$$

*Subcase 1.1.2:* Suppose that $X^- \neq \emptyset$ and $Y^- \neq \emptyset$. This case can be solved analogously to Subcase 1.1.1.

We consider three remaining cases.

*Subcase 1.1.3:* Suppose that $Y^+ \neq \emptyset$ and $Y^- \neq \emptyset$. By Subcases 1.1.1 and 1.1.2 it follows that $X^+ = X^- = \emptyset$ and thus, $\hat{X} \neq \emptyset$. Let $Y^+ = \{y_1\}$ and $Y^- = \{y_2\}$.

Let $D_1, D_2, \ldots, D_q$ be an out-branching of $D[\hat{X}]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y_2\}$ and thus, $|V(D_q)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(D_q)| \geq 2p$ if $D_q \not\rightarrow y_2$ by Lemma 1.31.(c). Let $C_q$ be a Hamiltonian cycle of $D_q$. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X|$ by choice of $C$. If $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_1)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$ (Subcase 1.1.3.1) and $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 1.1.3.2). Note that $D_q \rightarrow y_2$, since $|V(D_q)| = 2p - 1$.

*Subcase 1.1.3.1:* Suppose that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Note that $N^+(\hat{X}, C) = N^+(\hat{X}, Y^+) = \emptyset$ by Claims 1 and 3. It follows that $N^+(x, D_2) \neq \emptyset$ and hence $x \rightarrow D_2$. Since $N^-(\hat{X}, C) = \emptyset$ by Claim 1 and $\delta(D) \geq 2$, it follows that $\{y_1, y_2\} \rightarrow x$. Let

$$C' = xC_2y_2x.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(C') \cap X| \geq 4p.$$

*Subcase 1.1.3.2:* Suppose that $q = 1$ and $|V(D_1)| = 2p-1$. Note that $N^-(\hat{X}, C) = \emptyset$ by Claim 1. The latter together with Lemma 1.31.(b) implies that $y_1 \rightarrow D_1$.

*Subcase 1.1.3.2.1:* Suppose that $|N^-(y_1, C)| \geq 2$. We may assume, without loss of generality, that $\{v_i, v_k\} \rightarrow y_1$, where $1 \leq i \leq k - 1$. If $V(C[v_1, v_i]) - X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_1, v_i]) \cap X| \geq 2$. Let

$$C' = y_1C_1y_2C[v_1, v_i]y_1.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X|$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq |V(C') \cap X| + |V(D_1)| \geq 2|V(D_1)| + 2 = 4p.$$

So assume that $V(C[v_1, v_i]) \subseteq X$.

If $V(C) \not\subseteq X$, we consider

$$C^* = y_1C_1y_2C[v_1, v_i]y_1.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq |V(D_1)| + |V(C[v_1, v_i])|$ by choice of $C$. Hence

$$\begin{aligned} |X| &\geq |V(C) \cap X| + |V(D_1)| \\ &\geq 2|V(D_1)| + |V(C[v_1, v_i])| \\ &= 4p - 2 + |V(C[v_1, v_i])|. \end{aligned}$$

So it remains to consider the case that $p \geq 3$ and $i = 1$. If $v_k \in X$, let

$$\hat{C} = y_1 C_1 y_2 v_1 v_k y_1.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So assume that $v_k \notin X$. Since $\delta(D) \geq 3$, there exists an arc $v_j v_1$ in $D$, where $j \neq k$. Then either $|V(C[v_1, v_j]) \cap X| \geq 2$ and $V(C[v_{j+1}, v_k]) - X \neq \emptyset$ or $|V(C[v_{j+1}, v_k]) \cap X| \geq 2$ and $V(C[v_1, v_j]) - X \neq \emptyset$. We may assume, without loss of generality, that the former holds. Let

$$\tilde{C} = y_1 C_1 y_2 C[v_1, v_j] y_1.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

If $V(C) \subseteq X$, note that $D_1$ induces a $(p-1)$-regular tournament in $D$. Furthermore, note that $N^+(C) - V(C) \subseteq \{y_1\}$. So $|V(C)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(C)| \geq 2p$ if $C \not\rightarrow y_1$ by Lemma 1.31.(c). If $|V(C)| = 2p - 1$, it follows that $C \rightarrow y_1$. Since $\delta(D) \geq p$, the cycle $C$ induces a $(p-1)$-regular tournament in $D$ and thus, $D$ is a member of $\mathcal{T}^*_{loc}$. If $|V(C)| = 2p$, the cycle $C$ induces a tournament $T$ in $D$ such that $p - 1 \leq \delta(T), \Delta(T) \leq p$. So if $C \rightarrow y_1$, the digraph $D$ is a local tournament and a member of $\mathcal{T}^{**}_{loc}$. If $C \not\rightarrow y_1$, the digraph $D$ is not a local tournament and with the help of Lemma 5.11 we conclude that $D$ is a member of $\mathcal{T}^*_{in}$.

*Subcase 1.1.3.2.2:* Suppose that $|N^-(y_1, C)| = 1$. We conclude that $y_2 \rightarrow y_1$ and $p = 2$. If $|V(C) \cap X| \geq 2p = 4$, it follows that $|X| \geq 4p - 1 = 7$. So assume that $|V(C) \cap X| = |V(D_1)| = 2p - 1 = 3$. Let, without loss of generality, $v_k$ be the negative neighbor of $y_1$ on $C$. If there exists a vertex $v_i \in V(C) \cap X$ such that $V(C[v_1, v_{i-1}]) - X \neq \emptyset$, we consider the cycle

$$C' = y_1 C_1 y_2 C[v_i, v_k] y_1.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq |V(D_1)| + 1 = 4$ by choice of $C$, a contradiction. So $V(C) \cap X = \{v_1, v_2, v_3\}$ and $v_i \notin X$ for every index $4 \leq i \leq k$. Note that $k \geq 4$. If $D$ has an arc $v_i v_j$, where $1 \leq i \leq 3$ and $5 \leq j \leq k$, we consider the cycle

$$C^* = y_1 C_1 y_2 v_i C[v_j, v_k] y_1.$$

Then $V(D)-V(C^*)-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(C^*)\cap X| \geq |V(D_1)|+1 = 4$ by choice of $C$, a contradiction. So $N^+(\{v_1, v_2, v_3\}) - \{v_1, v_2, v_3\} \subseteq \{v_4\}$ and thus, $\{v_1, v_2\} \to v_4$ and $v_3 \to v_1$. The latter implies that $v_k \to v_3$ and hence $k > 4$. Since $\{v_1, v_2, v_3\} \to v_4$, it follows that there is an arc $v_4v_j$, where $5 < j \leq k$. Let

$$\hat{C} = y_1C_1y_2C[v_1, v_4]C[v_j, v_k]y_1.$$

Then $V(D)-V(\hat{C})-X \neq \emptyset$ and thus, $|V(C)\cap X| \geq |V(\hat{C})\cap X| \geq |V(D_1)|+3 = 6$ by choice of $C$, the final contradiction.

*Subcase 1.1.4.* Suppose that $Y^+ \neq \emptyset$ and $Y^- = \emptyset$. In this case $X^+ = \emptyset$ and $X^- \neq \emptyset$ by Subcase 1.1.1. Let $Y^+ = \{y\}$. Note that $N^-(X^- \cup \hat{X}) - (X^- \cup \hat{X}) \subseteq \{y\}$. Let $D_1, D_2, \ldots, D_q$ be an out-branching of $D[X^- \cup \hat{X}]$, where $q \geq 1$. Then $N^-(D_1) - V(D_1) \subseteq \{y\}$ and thus, $|V(D_1)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(D_1)| \geq 2p$ if $y \not\to D_1$ by Lemma 1.31.(c). Let $C_1$ be a Hamiltonian cycle of $D_1$. Then $V(D) - V(C_1) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_1) \cap X|$ by choice of $C$. If $|V(D_1)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_1)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_1)| = 2p-1$ and $|V(D_2)| = 1$ (Subcase 1.1.4.1) and $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 1.1.4.2). Note that $y \to D_1$, since $|V(D_1)| = 2p - 1$.

*Subcase 1.1.4.1:* Suppose that $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$. Let $V(D_2) = \{x_2\}$. Since $d^+(x_2) \geq 2$, it follows that $D$ has an arc leading from $x_2$ to $C$. So $x_2 \in X^-$ and hence, $x_2 \to C$. Since $d^-(x_2) \geq 2$, it follows that $D$ has an arc leading from $D_1$ to $x_2$, say $x_1x_2$. If $x_2 \to y$, let

$$C' = yC_1[x_1^+, x_1]x_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p.$$

If $x_2 \not\to y$, the vertex $y$ has at least two negative neighbors on $C$. We may assume, without loss of generality, that $\{v_i, v_k\} \to y$, where $1 \leq i \leq k - 1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC_1[x_1^+, x_1]x_2C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p + 1.$$

*Subcase 1.1.4.2:* Suppose that $q = 1$ and $|V(D_1)| = 2p - 1$. The latter together with Lemma 1.31.(b) implies that every vertex of $D_1$ has at least one positive neighbor on $C$. So $D_1 \to C$. Recall that $y$ has at least $p \geq 2$ negative neighbors on $C$.

*Subcase 1.1.4.2.1:* Suppose that $p = 2$. Let, without loss of generality, $\{v_i, v_k\} \to y$, where $1 \leq i \leq k-1$. Since $V(C) - X \neq \emptyset$ and $|V(C) \cap X| \geq 2p - 1 = 3$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p = 4$ by choice of $C$. Hence

$$|X| = |V(D_1)| + |V(C) \cap X| \geq 4p - 1 = 7.$$

*Subcase 1.1.4.2.2:* Suppose that $p \geq 3$. We may assume, without loss of generality, that $\{v_i, v_j, v_k\} \to y$, where $1 \leq i < j \leq k-1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(D_1)| + |V(C) \cap X| \geq 4p.$$

*Subcase 1.1.5:* Suppose that $Y^+ = \emptyset$ and $Y^- \neq \emptyset$. In this case $X^- = \emptyset$ and $X^+ \neq \emptyset$ by Subcase 1.1.2. Let $Y^- = \{y\}$. Note that $N^+(X^+ \cup \hat{X}) - (X^+ \cup \hat{X}) \subseteq \{y\}$. Let $D_1, D_2, \ldots, D_q$ be an out-branching of $D[X^+ \cup \hat{X}]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y\}$ and thus, $|V(D_q)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(D_q)| \geq 2p$ if $D_q \not\to y$ by Lemma 1.31.(c). Let $C_q$ be a Hamiltonian cycle of $D_q$. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q) \cap X|$ by choice of $C$. If $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_q)| \geq 3$, it follows that $|X| \geq 4p$. So it remains to check the cases $q = 2$, $|V(D_2)| = 2p-1$ and $|V(D_1)| = 1$ (Subcase 1.1.5.1) and $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 1.1.5.2). Note that $D_q \to y$, since $|V(D_q)| = 2p - 1$.

*Subcase 1.1.5.1:* Suppose that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Since $d^+(x) \geq 2$, it follows that $D$ has an arc leading from $x$ to $D_2$. Hence $x \to D_2$.

If $y \to x$, let

$$C' = yxC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$. Hence

$$|X| = |V(C) \cap X| + |V(D_2)| + 1 \geq 4p.$$

If $y \not\to x$, the vertex $x$ has at least two negative neighbors on $C$. Let, without loss of generality, $\{v_i, v_k\} \to x$, where $1 \leq i \leq k-1$. Since $|V(C) \cap X| \geq 3$ and $V(C) - X \neq \emptyset$, we may assume, without loss of generality, that $|V(C[v_{i+1}, v_k]) \cap X| \geq 1$ and $V(C[v_1, v_i]) - X \neq \emptyset$. Let

$$C^* = yC[v_{i+1}, v_k]xC_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_2)| + 1 \geq 4p + 1.$$

*Subcase 1.1.5.2:* Suppose that $q = 1$ and $|V(D_1)| = 2p - 1$. The latter together with Lemma 1.31.(b) yields that every vertex of $D_1$ has at least one negative neighbor on $C$. Note that $N^-(D_1, C) \to D_1$. Let, without loss of generality, $v_k \to D_1$.

Assume that there exists a vertex $v_i \in V(C) - X$ such that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. Let

$$C' = yC[v_{i+1}, v_k]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_2)| \geq 4p.$$

So assume the contrary. That means that $C$ has the following structure. Let $i = \min\{j \mid v_j \notin X\}$ be the minimal index such that $v_i$ is not in $X$. Then there is at most one vertex on $C[v_{i+1}, v_k]$ that belongs to $X$, that is $|V(C[v_{i+1}, v_k]) \cap X| \leq 1$.

*Subcase 1.1.5.2.1:* Suppose that $i = k$. Then $V(C) - X = \{v_k\}$.

If $N^-(D_1, C) \neq \{v_1, v_k\}$, let $v_j \notin \{v_1, v_k\}$ be a vertex that dominates $D_1$ and let

$$C' = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p.$$

If $N^-(D_1, C) = \{v_1, v_k\}$, let
$$C^* = yv_1C_1y.$$
Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq |V(D_1)| + 1 = 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p - 1$$

and it remains to consider the case that $p \geq 3$. In this case $v_1$ has a negative neighbor $v_r \neq v_k$ on $C$. Let
$$\hat{C} = yv_rv_1C_1y.$$
Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq |V(D_1)| + 2 = 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $N^-(D_1, C) = \{v_k\}$. Since $N^+(V(C) \cap X) - X \subseteq \{v_k\}$ and $v_k \to v_1 \in X$, it follows that $|V(C) \cap X| \geq 2p$ by Lemma 1.31.(c). So if $|X| = 4p - 1$ we conclude by Lemma 5.11 that $D$ is a member of $\mathcal{T}_{in}^{**}$.

*Subcase 1.1.5.2.2:* Suppose that $i \neq k$ and $V(C[v_{i+1}, v_k]) \cap X = \emptyset$. We conclude that $|V(C) - X| \geq 2$.

If $D$ has an arc $v_jv_k$, where $j \neq k - 1$, let

$$C' = yC[v_1, v_j]v_kC_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 1$ by choice of $C$. Hence
$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $N^-(v_k, C) = \{v_{k-1}\}$ which implies that $p = 2$. It suffices to show that $|V(C) \cap X| \geq 4$. So assume to the contrary that $V(C) \cap X = \{v_1, v_2, v_3\}$ and $i = 4$. Note that $N^+(\{v_1, v_2, v_3\}, C) - \{v_1, v_2, v_3\} \subseteq \{v_4\}$.

If $v_j \to D_1$ for an index $j \in \{1, 2, 3\}$, let

$$C^* = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 4$ by choice of $C$, a contradiction.

So $N^+(\{v_1, v_2, v_3\}) - \{v_1, v_2, v_3\} \subseteq \{v_4\}$. It follows that $v_3 \to v_1$ and $\{v_1, v_2, v_3\} \to v_4$. Then $v_4$ has a positive neighbor $v_j$ on $C$, where $j \geq 6$. Let

$$\hat{C} = yC[v_1, v_4]C[v_j, v_k]C_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 6$, again a contradiction.

*Subcase 1.1.5.2.3:* Suppose that $i \neq k$ and $V(C[v_{i+1}, v_k]) \cap X = \{x\}$.

Assume that $x \neq v_k$. If $N^-(D_1, C) \neq \{v_k\}$, let $v_j \to D_1$, where $j \neq k$. If $1 \leq j \leq i-1$, let

$$C' = yC[x, v_j]C_1y$$

and if $i \leq j \leq k-1$, let

$$C' = yC[v_1, v_j]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p+1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $N^-(D_1, C) = \{v_k\}$. We show that $p = 2$. If $v_{k-1} = x$, assume that $v_{k-1}$ has a negative neighbor $v_j \neq v_{k-2}$ on $C$. Let

$$C^* = yC[v_1, v_j]v_{k-1}v_kC_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p+1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $|N^-(v_{k-1}, C)| = 1$ and thus, $p = 2$. If $v_{k-1} \neq x$, assume that $v_k$ has a negative neighbor $v_j \neq v_{k-1}$ on $C$. Let

$$\hat{C} = yC[v_1, v_j]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p+1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $|N^-(v_k, C)| = 1$ and thus, $p = 2$. Therefore it remains to show that $|X| \geq 4p-1 = 7$. Let $A = V(C[v_1, v_{i-1}])$. Note that $A \subseteq X$ and $|V(C) \cap X| = |A| + 1$. If there is a vertex $v_j \in A$ that has a positive neighbor $v_r \in V(C) - A$ besides $v_i$, let

$$\tilde{C} = yC[v_1, v_j]C[v_r, v_k]C_1y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p = 4$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p - 1 = 7.$$

So $N^+(A) - A \subseteq \{v_i\}$ and thus, $|A| \geq 2p - 1 = 3$ by Lemma 1.31.(b). It follows that

$$|X| \geq |A| + 1 + |V(D_1)| \geq 4p - 1 = 7.$$

Assume that $x = v_k$. Let $A = V(C[v_1, v_{i-1}])$. Note that $A \subseteq X$ and $|V(C) \cap X| = |A| + 1$. If there is a vertex $v_j \in A$ that has a positive neighbor $v_r \in V(C) - A$ besides $v_i$, let

$$C' = yC[v_1, v_j]C[v_r, v_k]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

If there is a vertex $v_j \in A$ that dominates $D_1$, let

$$C^* = yC[v_k, v_j]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

So $N^+(A) - A \subseteq \{v_i\}$ and thus, $|A| \geq 2p - 1$ by Lemma 1.31.(b). Hence

$$|X| \geq |A| + 1 + |V(D_1)| \geq 4p - 1.$$

It remains to check the case $p \geq 3$. In this case $D$ has an arc $v_jv_k$, where $j \neq k-1$. Let

$$\hat{C} = yC[v_1, v_j]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| \geq 4p.$$

*Subcase 1.2.* Suppose that $|\hat{Y}| = 1$. Let $\hat{Y} = \{y\}$. Similar to the case $\hat{Y} = \emptyset$, we define

$$A = \{v \in V(D) \mid \text{there exists a path leading from } X^+ \cup Y^+ \text{ to } v \text{ in } D - y\}$$

and

$$B = \{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } X^- \cup Y^- \text{ in } D - y\}.$$

Note that $A \subseteq X^+ \cup Y^+ \cup \hat{X}$, $B \subseteq X^- \cup Y^- \cup \hat{X}$ and $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an out-branching of $D[A]$, where $q \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{y\}$ and thus, $|V(A_q)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(A_q)| \geq 2p$ if $A_q \not\rightarrow y$ by Lemma 1.31.(c). Analogously, let $B_1, B_2, \ldots, B_r$ be an out-branching of $D[B]$, where $r \geq 1$. Then $N^-(B_1) - V(B_1) \subseteq \{y\}$ and thus, $|V(B_1)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(B_1)| \geq 2p$ if $y \not\rightarrow B_1$ by Lemma 1.31.(c). Since $|V(A_q) \cap X| = |V(A_q) - Y^+| \geq 2p - 2$ and $|V(B_1) \cap X| = |V(B_1) - Y^-| \geq 2p - 2$, it follows that $|V(C) \cap X| \geq \max\{|V(A_q) \cap X|, |V(B_1) \cap X|\} \geq 2p - 2$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(A_q) \cap X| + |V(B_1) \cap X| \geq 6p - 6.$$

Since

$$6p - 6 \geq 4p \Leftrightarrow p \geq 3,$$

it remains to check the case $p = 2$. In this case we have to show that $|X| \geq 4p - 1 = 7$.

So assume to the contrary that $|X| \leq 4p - 2 = 6$. Then $|V(A_q) \cap X| = |V(B_1) \cap X| = |V(C) \cap X| = 2$, $V(A_q) = A$, $V(B_1) = B$, $X - A - B - V(C) = \emptyset$ and $A \rightarrow y \rightarrow B$. Furthermore, $V(C) - X \neq \emptyset$. Let $C_A = x_1x_2y_1x_1$ be a Hamiltonian cycle of $A$, where $\{y_1\} = Y^+$, and let $C_B = x_3x_4y_2x_3$ be a Hamiltonian cycle of $B$, where $\{y_2\} = Y^-$. If $D$ has an arc $uv$ from $B$ to $A$, let

$$C' = yu^+u^-uvv^+v^-y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 4$ by choice of $C$, a contradiction. So $D$ has no arc leading from $B$ to $A$. Since $\delta(D) \geq 2$, it follows that $N^+(u, C) \neq \emptyset$ and $N^-(v, C) \neq \emptyset$ for every vertex $u \in B$ and $v \in A$. Hence $B \rightarrow C$ by Claim 1. Let $w \in V(C)$ be a negative neighbor of $x_1$ and let

$$C^* = yx_3x_4y_2wx_1x_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 4$ by choice of $C$, again a contradiction.

*Subcase 1.3.* Suppose that $|\hat{Y}| \geq 2$. Note that $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$ by Claim 2. Let $P = z_0z_1 \ldots z_r$ be a shortest $(X^+ \cup Y^+)$-$(X^- \cup Y^-)$-path in $D$. Then $z_0 \in X^+ \cup Y^+$, $z_r \in X^- \cup Y^-$ and $\{z_1, z_2, \ldots, z_{r-1}\} \subseteq \hat{X} \cup \hat{Y}$. Furthermore, $\hat{Y} \subseteq V(P)$ by Claim 6 and thus, $r \geq 3$. Let $i = \min\{s \mid z_s \in \hat{Y}\}$ be the smallest integer such that $z_i \in \hat{Y}$ and let $j = \max\{s \mid z_s \in \hat{Y}\}$ be the greatest integer such that $z_j \in \hat{Y}$. We consider the positive and negative neighborhood of $z_s \in V(P)$, where $1 \leq s \leq r - 1$. In view of Claims 1 and 2, we have $N^-(z_s, C) = \emptyset$ and in view of Claims 1-4 we have $N^+(z_s) \subseteq X^- \cup Y^- \cup \hat{X} \cup \hat{Y}$. Because of the choice of $P$, it follows that $N^-(z_s) \subseteq \hat{X} \cup \hat{Y} \cup X^- \cup Y^-$ for $s \geq 2$ and $N^+(z_s) \subseteq \hat{X} \cup \hat{Y}$ for $s \leq r - 2$.

If $z_s$ has a positive neighbor $z_t \in V(P)$ besides $z_{s+1}$, we conclude $t < s-2$ because of the choice of $P$. Using the in-tournament property of $D$ and the choice of $P$, it follows that $z_s \to z_0 \in X^+ \cup Y^+$, a contradiction to Claim 3 or 4. So $N^+(z_s, P) = \{z_{s+1}\}$.

If $z_s$ has a negative neighbor $z_t \in V(P)$ besides $z_{s-1}$, we conclude $t > s+2$ because of the choice of $P$. Due to the observations above, it follows that $t = r$. So $N^-(z_s, P) \subseteq \{z_{s-1}, z_r\}$.

Now we consider $D - z_i$ and $D - z_j$ and define the sets

$$A = \{v \in V(D) \mid \text{there exists a path leading from } z_{i-1} \text{ to } v \text{ in } D - z_i\}$$

and

$$B = \{v \in V(D) \mid \text{there exists a path leading from } v \text{ to } z_{j+1} \text{ in } D - z_j\}\,.$$

Note that $A \subseteq X^+ \cup Y^+ \cup V(P[z_0, z_{i-1}]) \cup (\hat{X} - V(P))$, $B \subseteq X^- \cup Y^- \cup V(P[z_{j+1}, z_r]) \cup (\hat{X} - V(P))$ and $A \cap B = \emptyset$ by Claim 6. Let $A_1, A_2, \ldots, A_q$ be an out-branching of $D[A]$, where $q \geq 1$, and let $B_1, B_2, \ldots, B_s$ be an out-branching of $D[B]$, where $s \geq 1$. Then $N^+(A_q) - V(A_q) \subseteq \{z_i\}$ and thus, $|V(A_q)| \geq 2p-1$ by Lemma 1.31.(b) and $|V(A_1)| \geq 2p$ if $A_q \not\to z_i$ by Lemma 1.31.(c). Analogously, $N^-(B_1) - V(B_1) \subseteq \{z_j\}$ and thus, $|V(B_1)| \geq 2p-1$ by Lemma 1.31.(b) and $|V(B_1)| \geq 2p$ if $z_j \not\to B_1$ by Lemma 1.31.(c). Since $|V(A_q) \cap X| = |V(A_q) - Y^+| \geq 2p-2$ and $|V(B_1) \cap X| = |V(B_1) - Y^-| \geq 2p-2$, it follows that $|V(C) \cap X| \geq \max\{|V(A_q) \cap X|, |V(B_1) \cap X|\} \geq 2p-2$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(A_q) \cap X| + |V(B_1) \cap X| \geq 6p - 6.$$

Since

$$6p - 6 \geq 4p \Leftrightarrow p \geq 3,$$

it remains to check the case $p = 2$. In this case we have to show that $|X| \geq 4p - 1 = 7$.

So assume to the contrary that $|X| \leq 4p - 2 = 6$. Then $|V(A_q) \cap X| = |V(B_1) \cap X| = |V(C) \cap X| = 2$, $V(A_q) = A$, $V(B_1) = B$, $X - A - B - V(C) = \emptyset$, $A \to z_i$ and $z_j \to B$. Furthermore, $V(C) - X \neq \emptyset$. Hence $Y^+, Y^- \neq \emptyset$, $X^+ \cup Y^+ \subseteq A$ and $X^- \cup Y^- \subseteq B$. Since $z_j \to B$ and $P$ has minimal length, it follows that $j = s-1$. Analogously, since $A \to z_i$ and $P$ has minimal length, it follows that $i = 1$. Let $A = \{x_1, x_2, y\}$, where $\{x_1, x_2\} \in X$ and $Y^+ = \{y\}$. Since $A$ induces a 3-cycle in $D$, we may assume, without loss of generality, that $D[A] = x_1 x_2 y x_1$. So $x_2 \notin \hat{X}$ by Claim 3. Hence $x_2 \in X^+$. Let $v$ be a negative neighbor of $x_2$ on $C$ and let

$$C' = x_2 P[z_i, z_r] C[v^+, v] x_2.$$

Then $V(D) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |V(C) \cap X| + 1$, a contradiction to the choice of $C$.

*Case 2.* Suppose that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- = \emptyset$. In this case $N^-(C) - V(C) \subseteq \hat{Y}$, since $N^+(\hat{X}, C) = \emptyset$ by Claim 1. Since $D$ is strong, it follows that $D$ has an arc from $\hat{Y}$ to $C$ and thus, $|\hat{Y}| = 1$ and $Y^+ = \emptyset$ by Claim 2. Let $\hat{Y} = \{y\}$. Since $y \notin Y^-$, it follows that $N^-(y, C) \neq \emptyset$.

If $\hat{X} \neq \emptyset$, note that $N^+(\hat{X}, C) = \emptyset$ by Claim 1 and $N^+(\hat{X}, X^+) = \emptyset$ by Claim 3. Since $N^-(y, C) \neq \emptyset$, it follows that $N^+(\hat{X}, y) = \emptyset$ and thus, $D$ is not strong, a contradiction. So $\hat{X} = \emptyset$.

Let $D_1, D_2, \ldots, D_q$ be an out-branching of $D[X^+]$, where $q \geq 1$. Then $N^+(D_q) - V(D_q) \subseteq \{y\}$ and thus, $|V(D_q)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(D_q)| \geq 2p$ if $D_q \not\rightarrow y$ by Lemma 1.31.(c). Let $C_q$ be a Hamiltonian cycle of $D_q$. Then $V(D) - V(C_q) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C_q)|$ by choice of $C$. So if $|V(D_q)| \geq 2p$ or if $q \geq 3$ or if $|V(D_{q-1})| \geq 3$, it follows that $|X| \geq 4p$. It remains to check the case that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$ (Subcase 2.1) and the case that $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 2.2). Note that in both cases $D_q \rightarrow y$, since $|V(D_q)| = 2p - 1$.

*Subcase 2.1.* Suppose that $q = 2$, $|V(D_2)| = 2p - 1$ and $|V(D_1)| = 1$. Let $V(D_1) = \{x\}$. Since $d^+(x) \geq 2$ and $N^+(x, C) = \emptyset$, it follows that $x$ has a positive neighbor in $D_2$. Hence $x \rightarrow D_2$. Since $d^-(x) \geq 2$, either $y \rightarrow x$ or $y$ has at least two positive neighbors on $C$.

*Subcase 2.1.1.* Suppose that $y \rightarrow x$. Let

$$C' = yxC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

*Subcase 2.1.2.* Suppose that $y \not\rightarrow x$. Then $|N^+(y, C)| \geq 2$ and $|N^-(x, C)| \geq 2$. We may assume, without loss of generality, that $v_k \rightarrow y \rightarrow v_1$. Since $D$ is an in-tournament, it follows subsequently that $v_k \rightarrow D_2$ and $v_k \rightarrow x$. Let $v_i$ be a second negative neighbor of $x$ on $C$, where $2 \leq i \leq k - 1$.

If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C' = yC[v_1, v_i]xC_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_{i+1}, v_k]) \subseteq X$ and thus, $V(C) - X \subseteq V(C[v_1, v_i])$. Let $v_j$ be a second positive neighbor of $y$ on $C$, where $2 \leq j \leq k-1$.

If $i+1 \leq j \leq k$, let

$$C^* = yC[v_j, v_k]xC_2y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $1 \leq j \leq i$.

If $V(C[v_1, v_{j-1}]) - X \neq \emptyset$, let

$$\hat{C} = yC[v_j, v_k]xC_2y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p+1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_1, v_{j-1}]) \subseteq X$. Let $r = \min\{s \mid v_s \notin X\}$. Then $r \leq i$. Let $A = \{v_1, v_2, \ldots, v_{r-1}\}$.

If there is a vertex $v_s \in A$ that has a positive neighbor $v_t$ on $C$ outside of $A \cup \{v_r\}$, let

$$\tilde{C} = yC[v_1, v_s]C[v_t, v_k]xC_2y,$$

if there is a vertex $v_s \in A$ that dominates $x$, let

$$\tilde{C} = yC[v_1, v_s]xC_2y$$

and if there is a vertex $v_s \in A$ that has a positive neighbor in $D_2$, let

$$\tilde{C} = yC[v_1, v_s]C_2y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_r\}$ and thus, $|A| \geq 2p-1$ by Lemma 1.31.(b). Since $v_k \notin A$ and $v_k \in X$, it follows that $|V(C) \cap X| \geq 2p$ and thus,

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

*Subcase 2.2.* Suppose that $q = 1$ and $|V(D_1)| = 2p-1$. In this case $|N^+(y, C)| \geq 2$. We consider two subcases depending on the structure of the negative neighborhood of $y$ on $C$.

*Subcase 2.2.1.* Suppose that there exist two vertices $v \neq w$ on $C$ such that $\{v, w\} \to y \to \{v^+, w^+\}$. Let, without loss of generality, $v = v_k$ and $w = v_i$, where $i \neq k$. Then $\{v_i, v_k\} \to D_1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) \cap X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) - X \neq \emptyset$.

If $|V(C[v_1, v_i]) \cap X| \geq 2$, let

$$C' = yC[v_1, v_i]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $|V(C[v_1, v_i]) \cap X| = 1$. Since $|V(C) \cap X| \geq 3$, it follows that $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. If $V(C[v_1, v_i]) - X \neq \emptyset$, let

$$C^* = yC[v_{i+1}, v_k]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_1, v_i]) \subseteq X$ and thus, $v_1 = v_i \in X$, again a contradiction.

*Subcase 2.2.2.* Suppose that there do not exist two vertices $v \neq w$ on $C$ such that $\{v, w\} \to y \to \{v^+, w^+\}$. Then we may assume, without loss of generality, that $v_k \to y \to \{v_1, v_2\}$. It follows that $v_k \to D_1$. If $v_1 \notin X$, let

$$C' = yC[v_2, v_k]C_1y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 2$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p + 1.$$

So $v_1 \in X$. Analogously we can show that $v_2 \in X$ if $p \geq 3$.

*Subcase 2.2.2.1.* Suppose that $N^-(D_1, C) \neq \{v_k\}$. Let $i$ be the minimal index such that $v_i \to D_1$.

*Subcase 2.2.2.1.1:* Suppose that $p = 2$. Then it suffices to show that $|X| \geq 4p-1 = 7$.

If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C^* = yC[v_1, v_i]C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p = 4$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

So assume that $V(C[v_{i+1}, v_k]) \subseteq X$. Let $j$ be the minimal index such that $v_j \notin X$. Then $j \leq i$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor $v_s$ on $C$ outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p = 4$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1 = 3$ by Lemma 1.31.(b). Since $v_k \notin A$ and $v_k \in X$, it follows that $|V(C) \cap X| \geq 2p = 4$ and thus,

$$|X| = |V(C) \cap X| + |X^+| \geq 4p - 1 = 7.$$

*Subcase 2.2.2.1.2:* Suppose that $p \geq 3$. We have to show that $|X| \geq 4p$. Due to our assumption we conclude that $y \rightarrow \{v_1, v_2, v_3\}$ and $\{v_1, v_2\} \subseteq X$.

Assume that $i \neq 1$. If $V(C[v_{i+1}, v_k]) - X \neq \emptyset$, let

$$C^* = yC[v_1, v_i]C_1 y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $V(C[v_{i+1}, v_k]) \subseteq X$. Let $j$ be the minimal integer such that $v_j \notin X$. Then $j \leq i$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor $v_s$ on $C$ outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 1.31.(b). If $i \neq s - 1$, we conclude that

$$|X| \geq |X^+| + |A| + |\{v_{k-1}, v_k\}| \geq 4p.$$

So $i = s - 1$. If $V(C[v_j, v_i]) \cap X \neq \emptyset$, it follows that

$$|X| \geq |X^+| + |A| + |\{v_k\}| + 1 \geq 4p.$$

So assume that $V(C[v_j, v_i]) \cap X = \emptyset$. Since $d^-(v_k) \geq p \geq 3$, the vertex $v_k$ has a negative neighbor $v_t \neq v_{k-1}$ on $C$. Let

$$\hat{C} = yC[v_1, v_t]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

Assume that $i = 1$. If $v_k \notin X$, let $v_t \neq v_k$ be a negative neighbor of $v_1$ on $C$. Let

$$C^* = yC[v_2, v_t]v_1C_1y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $v_k \in X$. Let $j$ be the minimal integer such that $v_j \notin X$ and let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor $v_s$ on $C$ outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 1.31.(b). If $V(C[v_j, v_{k-2}]) \cap X \neq \emptyset$, we conclude that

$$|X| \geq |X^+| + |A| + |\{v_k\}| + 1 \geq 4p.$$

So $V(C[v_j, v_{k-2}]) \cap X = \emptyset$. Since $d^-(v_k) \geq p \geq 3$, the vertex $v_k$ has a negative neighbor $v_t \neq v_{k-1}$ on $C$. Let

$$\hat{C} = yC[v_1, v_t]v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

*Subcase 2.2.2.2:* Suppose that $N^-(D_1, C) = \{v_k\}$. Let $j$ be the smallest index such that $v_j \notin X$.

*Subcase 2.2.2.2.1:* Suppose that $v_k \in X$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor $v_s$ on $C$ outside of $A \cup \{v_j\}$, let

$$C^* = yC[v_1, v_r]C[v_s, v_k].$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 1.31.(b). If $v_{k-1} \in X$, we conclude that

$$|X| \geq |A| + |X^+| + |\{v_{k-1}, v_k\}| \geq 4p.$$

So $v_{k-1} \notin X$. Since $d^-(v_k) \geq p \geq 3$, there is a negative neighbor $v_t \neq v_{k-1}$ of $v_k$ on $C$. Let

$$\tilde{C} = yC[v_1, v_t]v_k C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

*Subcase 2.2.2.2.2:* Suppose that $v_k \notin X$.

If $j < k$, we consider $v_{k-1}$. If $v_{k-1} \notin X$, let $v_t \neq v_{k-1}$ be a negative neighbor of $v_k$ on $C$. Let

$$C^* = yC[v_1, v_t]v_k y.$$

Then $V(D) - V(C^*) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C^*) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $v_{k-1} \in X$ and $j \leq k - 2$. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$. If there is a vertex $v_r \in A$ that has a positive neighbor $v_s$ on $C$ outside of $A \cup \{v_j\}$, let

$$\tilde{C} = yC[v_1, v_r]C[v_s, v_k]C_1 y.$$

Then $V(D) - V(\tilde{C}) - X \neq \emptyset$ and thus, $|V(\tilde{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^+(A) - A = \{v_j\}$ and thus, $|A| \geq 2p - 1$ by Lemma 1.31.(b). If $|V(C[v_j, v_k]) \cap X| \geq 2$, we conclude that

$$|X| \geq |A| + |X^+| + 2 = 4p.$$

So assume that $V(C[v_j, v_k]) \cap X = \{v_{k-1}\}$. If $v_{k-1}$ has a negative neighbor $v_t \neq v_{k-2}$ on $C$, let

$$\hat{C} = yC[v_1, v_t]v_{k-1}v_kC_1y.$$

Then $V(D) - V(\hat{C}) - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(\hat{C}) \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |X^+| \geq 4p.$$

So $N^-(v_{k-1}, C) = \{v_{k-2}\}$ and thus, $p = 2$. Since

$$|X| = |X^+| + |A| + 1 = 4p - 1 = 7,$$

there is nothing left to show.

If $j = s$, it follows that $|A| = |V(C) \cap X| \geq 2p$ by Lemma 1.31.(c). So if $|X| = 4p - 1$ we conclude by Lemma 5.11 that $D$ is a member of $\mathcal{T}_{in}^{**}$.

*Case 3.* Suppose that $X^- \cup Y^- \neq \emptyset$ and $X^+ \cup Y^+ = \emptyset$. Recall that $N^+(C, \hat{X}) = \emptyset$ by Claim 1. Since $D$ is strong, it follows that $\hat{Y} \neq \emptyset$ and $N^+(C, \hat{Y}) \neq \emptyset$. By Claim 2 we conclude that $|\hat{Y}| = 1$ and $Y^- = \emptyset$. The latter implies that $X^- \neq \emptyset$. Let $\hat{Y} = \{y\}$. Then $N^+(y, C) \neq \emptyset$ and $N^-(y, C) \neq \emptyset$. Since $D$ is an in-tournament, the vertex $y$ is adjacent to every vertex of $X^-$. Let $D_1, D_2, \ldots, D_q$ be an out-branching of $D[X^- \cup \hat{X}]$, where $q \geq 1$. Then $N^-(D_1) - V(D_1) \subseteq \{y\}$ and thus, $|V(D_1)| \geq 2p - 1$ by Lemma 1.31.(b) and $|V(D_1)| \geq 2p$ if $y \not\to D_1$ by Lemma 1.31.(c). Let $C_1$ be a Hamiltonian cycle of $D_1$. Since $V(D) - V(C_1) - X \neq \emptyset$, it follows that $|V(C) \cap X| \geq |V(C_1) \cap X| = |V(D_1)|$ by choice of $C$. If $|V(D_1)| \geq 2p$ or if $q \geq 3$ or if $q = 2$ and $|V(D_2)| \geq 3$, we conclude that $|X| \geq 4p$. It remains to check the case $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$ (Subcase 3.1) and the case $q = 1$ and $|V(D_1)| = 2p - 1$ (Subcase 3.2). Note that in both cases $y \to D_1$, since $|V(D_1)| = 2p - 1$.

*Subcase 3.1.* Suppose that $q = 2$, $|V(D_1)| = 2p - 1$ and $|V(D_2)| = 1$. Let $V(D_2) = \{x_2\}$. Since $x_2 \in X^- \cup \hat{X}$, it follows that $N^-(x_2, C) = \emptyset$. The latter together with $d^-(x_2) \geq p \geq 2$ yields that there is a vertex $x_1 \in V(D_1)$ that dominates $x_2$.

*Subcase 3.1.1.* Suppose that $x_2 \to y$. Let

$$C' = yC_1[x_1^+, x_1]x_2y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| = 2p$ by choice of $C$. Hence

$$|X| \geq |V(C) \cap X| + |V(D_1)| + 1 \geq 4p.$$

*Subcase 3.1.2.* Suppose that $x_2 \not\to y$. Then $|N^-(y, C)| \geq 2$. In addition, note that $N^+(x_2, C) \neq \emptyset$ and thus, $x_2 \to C$. Let, without loss of generality, $\{v_i, v_k\} \to y$, where $i \leq k-1$. We may assume, without loss of generality, that $V(C[v_1, v_i])-X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) \cap X \neq \emptyset$. Let

$$C' = yC_1[x_1^+, x_1]x_2C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| + 1 \geq 4p + 1.$$

*Subcase 3.2:* Suppose that $q = 1$ and $|V(D_1)| = 2p - 1$. Then $D_1 \to C$ and $|N^-(y, C)| \geq p$.

*Subcase 3.2.1:* Suppose that $|N^-(y, C)| \geq 3$. Let, without loss of generality, $\{v_i, v_j, v_k\} \to y$, where $1 \leq i < j \leq k - 1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) - X \neq \emptyset$ and $|V(C[v_{i+1}, v_k]) \cap X| \geq 2$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p + 1$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p.$$

*Subcase 3.2.2:* Suppose that $|N^-(y, C)| = p = 2$. We have to show that $|X| \geq 4p - 1 = 7$. Let, without loss of generality, $\{v_i, v_k\} \to y$, where $i \leq k - 1$. We may assume, without loss of generality, that $V(C[v_1, v_i]) - X \neq \emptyset$ and $V(C[v_{i+1}, v_k]) \cap X \neq \emptyset$. Let

$$C' = yC_1C[v_{i+1}, v_k]y.$$

Then $V(D) - V(C') - X \neq \emptyset$ and thus, $|V(C) \cap X| \geq |V(C') \cap X| \geq 2p = 4$ by choice of $C$. Hence

$$|X| = |V(C) \cap X| + |V(D_1)| \geq 4p - 1$$

which completes the proof of this lemma. □

**Lemma 5.17.** *If $X \subseteq V(C)$, then $V(D) - V(C) = \hat{Y}$ and $|\hat{Y}| = 1$.*

*Proof.* We consider five cases depending on $Y$.

*Case 1.* Suppose that $|Y^+| = |Y^-| = 1$ and $\hat{Y} \neq \emptyset$. Since $D$ is strong, there exists an $Y^+$-$Y^-$-path in $D$. Let $P = y_0y_1 \ldots y_s$ be a shortest such path. It follows that $V(P) = Y$ by Claim 6. But then, since $y_0 \in Y^+$ and because of the choice of $P$, it follows that $N^+(y_0) = \{y_1\}$ which contradicts $\delta^+(D) \geq 2$.

*Case 2.* Suppose that $|Y^+| = |Y^-| = 1$ and $\hat{Y} = \emptyset$. Let $Y^+ = \{y_1\}$ and $Y^- = \{y_2\}$. Then $d^+(y_1), d^-(y_2) \leq 1$, a contradiction.

*Case 3.* Suppose that $|Y^+| = 1$, $Y^- = \emptyset$ and $\hat{Y} \neq \emptyset$ or $|Y^-| = 1$, $Y^+ = \emptyset$ and $\hat{Y} \neq \emptyset$. By Claim 4 $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$ and thus, $D$ is not strong, a contradiction.

*Case 4.* Suppose that $|Y^+| = 1$ and $Y^- = \hat{Y} = \emptyset$ or $|Y^-|$ and $Y^+ = \hat{Y} = \emptyset$. It follows that $D$ is not strong, again a contradiction.

*Case 5.* Suppose that $Y^+ = Y^- = \emptyset$ and $\hat{Y} \neq \emptyset$. Since $D$ is strong, we conclude that $N^+(\hat{Y}, C) \neq \emptyset$ and $N^-(\hat{Y}, C) \neq \emptyset$. By Claim 4 it follows that $|\hat{Y}| = 1$ which completes the proof of this lemma. □

## 5.4 Generalizations of Kotani's Theorems

In this section we use the results of Section 5.3 to generalize Theorems 5.7, 5.8 and 5.9 to local tournaments and in-tournaments. The following three results summarize our present achievements.

**Theorem 5.18.** *Let $p \geq 2$ be an integer and let $D$ be a strong in-tournament with $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p - 3$ and $X \neq V(D)$, then there exists a cycle $C$ in $D$ such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$.*

*Proof.* Let $X$ be a subset of $V(D)$ such that $|X| \leq 4p-3$ and $X \neq V(D)$. Let $C$, $X^+$, $X^-$, $\hat{X}$, $X$, $Y^+$, $Y^-$, $\hat{Y}$ and $Y$ be defined as above.

Assume that $X - V(C) \neq \emptyset$. If $|X| = |V(D)| - 1$, then we get a contradiction by Lemma 5.15. If $|X| \leq |V(D)| - 2$, then we get a contradiction by Lemma 5.16.

Hence, we obtain $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ by Lemma 5.17. □

**Theorem 5.19.** *Let $p \geq 2$ be an integer and let $D$ be a strong in-tournament with $|V(D)| \geq 4p$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p - 2$, then either*

*(a) there exists a cycle $C$ in $D$ such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or*

*(b) $D \in \mathcal{T}^*_{loc}$.*

*Proof.* Let $X$ be a subset of $V(D)$ such that $|X| \leq 4p-2$ and let $C$, $X^+$, $X^-$, $\hat{X}$, $X$, $Y^+$, $Y^-$, $\hat{Y}$ and $Y$ be defined as above. In addition, let $D \notin \mathcal{T}^*_{loc}$.

Assume that $X - V(C) \neq \emptyset$. Since $D \notin \mathcal{T}^*_{loc}$, Lemma 5.16.(a) implies that $|X| \geq 4p - 1$, a contradiction to our assumption. Hence, by Lemma 5.17, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

**Theorem 5.20.** *Let $p \geq 3$ be an integer. Let $D$ be a strong in-tournament with $|V(D)| \geq 4p+1$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p-1$, then either*

*(a) there exists a cycle $C$ in $D$ such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or*

*(b) $D \in \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$.*

*Proof.* Let $X$ be a subset of $V(D)$ such that $|X| \leq 4p-1$ and let $C$, $X^+$, $X^-$, $\hat{X}$, $X$, $Y^+$, $Y^-$, $\hat{Y}$ and $Y$ be defined as above. In addition, let $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$.

Assume that $X - V(C) \neq \emptyset$. Note that $D \notin \mathcal{T}^*_{loc}$, since $|V(D)| \geq 4p+1$. Additionally $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ and thus, Lemma 5.16.(b) implies that $|X| \geq 4p$, a contradiction to our assumption. Hence, by Lemma 5.17, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

The combination of Lemma 5.10 and the theorems above yields the following results.

**Theorem 5.21.** *Let $D$ be a strong in-tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then $D$ has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Theorem 5.22.** *Let $D$ be a strong in-tournament such that $D \notin \mathcal{T}^*_{loc}$ and let $p \geq 2$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p$, then $D$ has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Theorem 5.23.** *Let $D$ be a strong in-tournament such that $D \notin \mathcal{T}^*_{in} \cup \mathcal{T}^{**}_{in} \cup \mathcal{T}^{**}_{loc}$ and let $p \geq 3$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p+1$, then $D$ has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

For local tournaments we can state these results as follows.

**Corollary 5.24** (Meierling & Volkmann [55] 2007)**.** *Let $D$ be a strong local tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then $D$ has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Corollary 5.25** (Meierling & Volkmann [55] 2007)**.** *Let $D$ be a strong local tournament such that $D \notin \mathcal{T}^*_{loc}$ and let $p \geq 2$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p$, then $D$ has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

**Corollary 5.26** (Meierling & Volkmann [55] 2007)**.** *Let $D$ be a strong local tournament such that $D \notin \mathcal{T}_{loc}^{**}$ and let $p \geq 3$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p+1$, then $D$ has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Since the exceptional classes of in-tournaments and local tournaments do not contain any tournaments, Theorems 5.7, 5.8 and 5.9 by Kotani [50] are direct consequences of our results.

# Chapter 6

# Non-separating vertices in local tournaments with low degree

In Chapter 5 we considered the following problem: how many non-separating vertices does a strongly connected in-tournament have at the least if the minimum degree $\delta$ is greater or equal than $p \geq 2$? We showed that the answer is $k = \min\{|V(T)|, 4p-2\}$ (cf. Theorem 5.21). In this chapter we discuss the same question for strongly connected local tournaments with no restrictions on the minimum degree.

In 1990, Bang-Jensen [3] proved that every strongly connected locally semicomplete digraph that is not a cycle has at least one non-separating vertex (cf. Theorem 5.3). Four years later, Guo and Volkmann [36] improved this result in showing that every strongly connected locally semicomplete digraph on $n \geq 4$ vertices has at least two non-separating vertices if it has at least $n+2$ arcs and determined the digraph in Figure 5.1 to be the only locally semicomplete digraph with exactly one non-separating vertex (cf. Theorem 5.4). We reformulate Theorem 5.4.(a) in terms of vertex degree.

**Theorem 6.1** (Guo & Volkmann [36] 1994)**.** *Let $D$ be a strong locally semicomplete digraph. If $D$ has two vertices $y_1, y_2$ such that $d^+(y_i) \geq 2$ for $i = 1, 2$ or $d^-(y_i) \geq 2$ for $i = 1, 2$, then $D$ has at least two non-separating vertices.*

Using Moon's Theorem [57] (cf. Theorem 2.3), Korvin [49] showed already in 1967 that every strongly connected tournament has at least two non-separating vertices (cf. Corollary 5.1). In 1975, Las Vergnas [51] characterized all strongly connected tournaments with exactly two non-separating vertices to be isomorphic to the tournament $Q_n$ (cf. Theorem 5.2). In this chapter we characterize all strongly connected local tournaments that have exactly two non-separating vertices thereby

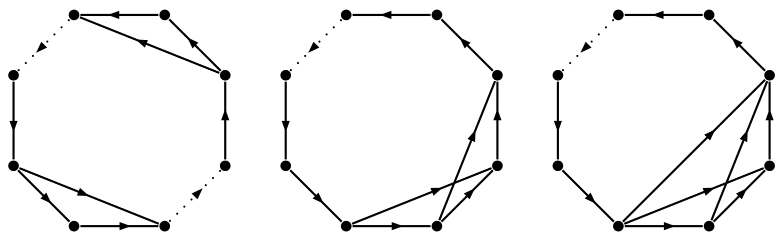

Figure 6.1: Local tournaments $C_n^j$ with exactly two non-separating vertices.

generalizing Theorem 5.2 for tournaments. Firstly we would like to remark the following.

**Remark 6.2.** *The tournament $Q_n$ has only two non-separating vertices, the vertices $x_1$ and $x_n$. So every local tournament $D$ on $n$ vertices that is a subdigraph of $Q_n$ has at most two non-separating vertices.*

## 6.1 The exceptional cases

In this section we present two classes of examples that demonstrate the sharpness of Theorem 6.1 (cf. Figure 6.1).

**Definition 6.3.** *Let $C = x_1x_2 \dots x_nx_1$ be a cycle on $n$ vertices. Let*

*(a) $C_n^j$ be the strong local tournament that consists of the cycle $C$ and the additional arcs $x_1x_3$ and $x_jx_{j+2}$, where $j \in \{2, 3, \dots, n\}$ ;*

*(b) $C_n^1$ be the strong local tournament that consists of the cycle $C$ and the additional arcs $x_1x_3$, $x_1x_4$ and $x_2x_4$ .*

It is easy to see that the only non-separating vertices of $C_n^j$ (for $j = 2, 3, \dots, n$) are $x_2$ and $x_{j+1}$ and the only non-separating vertices of $C_n^1$ are $x_2$ and $x_3$. The local tournaments $C_n^j$ show that Theorem 6.1 is sharp in terms of number of arcs and the local tournament $C_n^1$ shows that Theorem 6.1 is sharp in terms of vertex degree.

In the next example we present a local tournament that is not a tournament and has exactly two non-separating vertices, but is not isomorphic to $C_n^j$ for $j = 1, 2, \dots, n$ (cf. Figure 6.2).

**Definition 6.4.** *Let $Q_5^*$ be the local tournament $Q_5 - x_4x_2$, where $Q_5$ is defined as in Theorem 5.2.*

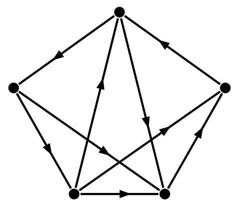

Figure 6.2: The local tournament $Q_5^*$ with exactly two non-separating vertices.

## 6.2 Local tournaments with exactly two non-separating vertices

In this section we characterize all local tournaments that have exactly two non-separating vertices to be the exceptional cases of Section 6.1. In view of Theorem 5.4.(a) we focus on local tournaments $D$ with at least $|V(D)| + 2$ arcs.

**Theorem 6.5.** *Let $D$ be a strong local tournament on $n$ vertices with at least $n + 2$ arcs. Then $D$ has exactly two non-separating vertices if and only if $D$ is isomorphic to $Q_5^*$, $Q_n$ or $C_n^j$ for an integer $1 \leq j \leq n$.*

*Proof.* It is easy to check that the local tournaments $Q_n$, $Q_5^*$ and $C_n^j$ for $j = 1, 2, \ldots, n$ have only two non-separating vertices.

So let $D$ be a strong local tournament on $n$ vertices with at least $n + 2$ arcs that is neither isomorphic to $Q_n$ nor to $Q_5^*$ nor to $C_n^j$ for $j = 1, 2, \ldots, n$. We shall show by induction on $n$ that $D$ has at least three non-separating vertices. Note that $n \geq 4$, since $D$ has at least $n + 2$ arcs.

**Induction basis.** The only strong local tournament on $n = 4$ vertices is isomorphic to $Q_4$. So the proposition is valid for $n = 4$.

**Inductive step.** Let $n \geq 5$ be an integer. Let $D$ be a strong local tournament on $n$ vertices with at least $n + 2$ arcs that is neither isomorphic to $Q_n$ nor to $Q_5^*$ nor to $C_n^j$ for $j = 1, 2, \ldots, n$. Observe that $D$ has at least $n + 3$ arcs, since $D$ is not isomorphic to $C_n^j$ for $j = 2, 3, \ldots, n$. In addition, since $D$ is not isomorphic to $C_n^1$, it follows that $D$ has at least three vertices with out-degree at least two.

If $D$ is 2-connected, the local tournament $D$ does not have any separating vertices. So assume that $D$ has a separating vertex $s$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - s$, where $p \geq 2$. Note that, according to Theorem 1.6, the vertex $s$ has an out-neighbor in $D_1$ and an in-neighbor in $D_p$. By Lemma 1.12 it follows that either $s \to D_1$ or that $s$ has positive as well as negative neighbors in $D_1$. Analogously, either $D_p \rightarrow s$ or $s$ has negative as well as positive neighbors in $D_p$. We distinguish the following cases.

*Case 1:* Suppose that $D_p \rightarrow s \rightarrow D_1$.

If $D - s$ has a strong component $D_i$ with at least three vertices, we conclude that $D - v$ is strong for every vertex $v$ of $D_i$. Hence $D$ has at least three non-separating vertices.

So assume that every strong component of $D - s$ consists of exactly one vertex. Let $V(D_i) = \{x_i\}$ for $i = 1, 2, \ldots, p = n - 1$. If $d^+(x_i) \geq 2$ for an index $i < n - 1$, we conclude by Theorem 1.6 that $x_i \rightarrow x_{i+2}$ (or $x_{n-2} \rightarrow s$ if $i = n - 2$). Hence $D - x_{i+1}$ is strong. Since $D$ has at least three vertices with out-degree at least two, we obtain that at least two vertices of $\{x_2, x_3 \ldots, x_{n-1}\}$ are non-separating vertices of $D$. Now we consider the vertex $s$.

If $d^+(s) = 1$, there exist three vertices $x_i$, $x_j$, $x_k$ with out-degree at least two. Due to our observations above $x_{i+1}$, $x_{j+1}$ and $x_{k+1}$ are non-separating vertices of $D$.

So assume that $d^+(s) \geq 2$. Then either $s \rightarrow x_2$ or there exists an index $2 \leq i \leq n - 3$ such that $x_i \rightarrow s \rightarrow x_{i+1}$. In the first case $D - x_1$ is strong and we are done. In the second case we distinguish the cases $n = 5$ and $n \geq 6$.

If $n = 5$, we conclude that $i = 2$ and the local tournament $D$ is either isomorphic to $Q_5$ or to $Q_5^*$, a contradiction.

So assume that $n \geq 6$. If $i = 2$, we conclude that $x_1 \rightarrow x_3$ and that $x_2 \rightarrow \{x_4, x_5, \ldots, x_{n-1}\}$. The latter implies particularly that $x_3 \rightarrow x_5$ and thus, $x_2$, $x_3$ and $x_4$ are non-separating vertices of $D$. So assume that $i \geq 3$. In this case $x_1 \rightarrow \{x_3, x_4, \ldots, x_{i+1}\}$ and $x_i \rightarrow \{x_{i+2}, x_{i+3}, \ldots, x_{n-1}\}$. The former implies particularly that $x_2 \rightarrow x_4$ and therefore $x_2$, $x_3$ and $x_{i+1}$ are non-separating vertices of $D$. This completes the proof of Case 1.

*Case 2:* Suppose that $s \not\rightarrow D_1$ or $D_p \not\rightarrow s$. So $s$ has in-neighbors in $D_1$ and $D_p$ or out-neighbors in $D_1$ and $D_p$. It follows by Theorem 1.6 that $D_i \rightarrow D_j$ for all indices $i < j$. Hence $D - v$ is strong for every vertex $v$ in $D_i$ with $2 \leq i \leq p - 1$.

If $|V(D_1)| \geq 3$, let $C_1$ be a Hamiltonian cycle of $D_1$ and let $x_1^+$ be an out-neighbor of $s$ in $D_1$. Then $C_1[x_1^+, x_1^-]$ is a Hamiltonian path of $D_1 - x_1$ and, since $D_1 \rightarrow D_2$, it is easy to see that $D - x_1$ is strong. It follows that the number of non-separating vertices of $D$ in $D_1$ is at least $|N^+(s, D_1)|$. Analogously we can show that if $|V(D_p)| \geq 3$, the number of non-separating vertices of $D$ in $D_p$ is at least $|N^-(s, D_p)|$.

If $|V(D_1)| \geq 3$ and $x \neq y$ are two vertices of $D_1$ such that $x$ is an out-neighbor of $s$ and $D_1 - y$ is strong, the vertex $y$ is a non-separating vertex of $D$. Analogously if

$|V(D_p)| \geq 3$ and $v \neq w$ are two vertices of $D_p$ such that $v$ is an in-neighbor of $s$ and $D_p - w$ is strong, the vertex $w$ is a non-separating vertex of $D$.

Since $s \not\to D_1$ or $D_p \not\to s$, we may assume, without loss of generality, that $|V(D_1)| \geq 3$. Recall that there is at least one non-separating vertex of $D$ in $D_1$ and that every vertex $v$ in $D_i$ with $2 \leq i \leq p-1$ is a non-separating vertex. So we assume that $p \leq 3$ and $\sum_{i=2}^{p-1} |V(D_i)| \leq 1$. It remains to consider the cases $p = 3$ and $|V(D_2)| = |V(D_3)| = 1$ (Subcase 2.1), $p = 2$ and $|V(D_2)| \geq 3$ (Subcase 2.2) and $p = 2$ and $|V(D_2)| = 1$ (Subcase 2.3).

*Subcase 2.1:* Suppose that $p = 3$ and $|V(D_2)| = |V(D_3)| = 1$. Let $V(D_2) = \{v\}$ and $V(D_3) = \{w\}$. Observe that $D - v$ is strong, since $D_1 \to w$.

Hence, if $|N^+(s, D_1)| \geq 2$, two vertices of $D_1$ and $v$ are non-separating vertices of $D$. So assume that $N^+(s, D_1) = \{u\}$.

If $v \to s$, the vertices $v$, $w$ and at least one vertex of $D_1$ are non-separating vertices of $D$. Hence $s \to v$. Since $D_1 \to D_2$ and $s \to v$, it follows that $(D_1 - u) \to s$.

If $|V(D_1)| = 3$, it is easy to check that $D$ is isomorphic to $Q_6$, a contradiction. Therefore $|V(D_1)| \geq 4$.

Note that $D_1$ is a strong subtournament of $D$ on at least four vertices. By Corollary 5.1 the component $D_1$ has at least two non-separating vertices $x$ and $y$. If $u \notin \{x, y\}$, the vertices $x$, $y$ and $v$ are non-separating vertices of $D$. So assume that $D_1$ has exactly two non-separating vertices and $u$ is one of them.

Let $u_1 u_2 \ldots u_{n-3} u_1$ be a Hamiltonian cycle of $D_1$. By the induction hypothesis we conclude that $D_1$ is isomorphic to $Q_{n-3}$ and $u \in \{u_1, u_{n-3}\}$. If $u = u_{n-3}$, the vertices $u_1$, $u_{n-4}$ and $v$ are non-separating vertices of $D$. Hence $u = u_1$. But then $D$ is isomorphic to $Q_n$ as defined in Theorem 5.2 with $v = x_1$, $w = x_2$, $s = x_3$ and $x_{i+3} = u_i$ for $i = 1, 2, \ldots, n-3$, a contradiction.

*Subcase 2.2:* Suppose that $p = 2$ and $|V(D_2)| \geq 3$. If $|N^+(s, D_1)| \geq 2$ (or $|N^-(s, D_2)| \geq 2$), two vertices of $D_1$ (of $D_2$) and a vertex of $D_2$ (of $D_1$) are non-separating vertices of $D$. So assume that $N^+(s, D_1) = \{u\}$ and $N^-(s, D_2) = \{v\}$. Since $N^+(s, D_2) \neq \emptyset \neq N^-(s, D_1)$, it follows that $(D_1 - u) \to s \to (D_2 - v)$. Let $u_1 u_2 \ldots u_r u_1$ be a Hamiltonian cycle of $D_1$ and let $v_1 v_2 \ldots v_t v_1$ be a Hamiltonian cycle of $D_2$.

If $r = 3$, note that $D_1$ is isomorphic to $Q_3$. We may assume, without loss of generality, that $\{u_2, u_3\} \to s \to u_1$. Analogously, if $t = 3$, the component $D_2$ is isomorphic to $Q_3$ and we may assume, without loss of generality, that $v_3 \to s \to \{v_1, v_2\}$.

Assume that $r \geq 4$. Then the component $D_1$ has at least two non-separating vertices $\{x, y\}$ by Corollary 5.1. If $u \notin \{x, y\}$, the vertices $x$, $y$ and a vertex of $D_2$ are non-separating vertices of $D$. Hence $D_1$ has exactly two non-separating vertices and $u$ is one of them. By the induction hypothesis we conclude that $D_1$ is isomorphic to $Q_r$ and $y \in \{u_1, u_r\}$. If $u = u_r$, the vertices $u_1$, $u_{r-1}$ and a vertex of $D_2$ are non-separating vertices of $D$. Therefore $u = u_1$. Analogously, if $t \geq 4$, the component $D_2$ is isomorphic to $Q_t$ with vertices $v_1, v_2, \ldots, v_t$ and $v = v_t$.

But then $D$ is isomorphic to $Q_n$ as defined in Theorem 5.2 with $x_i = v_i$ for $i = 1, 2, \ldots, t$, $x_{i+1} = s$ and $x_{i+2} = u_i$ for $i = 1, 2, \ldots, r$, a contradiction.

*Subcase 2.3:* Suppose that $p = 2$ and $|V(D_2)| = 1$. Then the single vertex $v$ of $D_2$ is a non-separating vertex of $D$.

If $|N^+(s, D_1)| \geq 2$, two vertices of $D_1$ and $v$ are non-separating vertices of $D$. So assume that $N^+(s, D_1) = \{u\}$. It follows that $(D_1 - u) \rightsquigarrow s$. Let $u_1 u_2 \ldots u_{n-2} u_1$ be a Hamiltonian cycle of $D_1$.

If $|V(D_1)| = 3$, note that $D_1$ is isomorphic to $Q_3$. Moreover, $n = 5$ and we may assume, without loss of generality, that $u_3 \rightarrow u_1$. If $\{u_2, u_3\} \rightarrow s \rightarrow u_1$, the local tournament $D$ is isomorphic to $Q_5$ and if $u_2$ and $s$ are not adjacent, the local tournament $D$ is isomorphic to $Q_5^*$, both cases contradict our assumption.

So assume that $|V(D_1)| \geq 4$. In view of Corollary 5.1, the component $D_1$ has at least two non-separating vertices $x, y$. If $u \notin \{x, y\}$, the vertices $x$, $y$ and $v$ are non-separating vertices of $D$. Hence $D_1$ has exactly two non-separating vertices and $u$ is one of them. By the induction hypothesis we conclude that $D_1$ is isomorphic to $Q_{n-2}$ and $u \in \{u_1, u_{n-2}\}$. If $u = u_{n-2}$, the vertices $u_1$, $u_{n-3}$ and $v$ are non-separating vertices of $D$. Therefore $u = u_1$. It follows that $(D_1 - u) \rightarrow s$. But then $D$ is isomorphic to $Q_n$ as defined in Theorem 5.2 with $z = x_1$, $s = x_2$ and $x_{i+2} = y_i$ for $i = 1, 2, \ldots, n-2$, a contradiction. This completes the proof of this theorem. □

In view of Theorem 5.4, Theorem 5.1 is an immediate corollary of the above theorem.

# Part III

# Cycles in local tournaments and in-tournaments

# Chapter 7

# The number of cycles in local tournaments

In Chapter 6 we investigated the number of non-separating vertices of local tournaments and characterized all local tournaments on $n$ vertices with at least $n+2$ arcs that have exactly two such vertices. In other words we characterized all local tournaments on $n$ vertices with at least $n+2$ arcs that have exactly two cycles of length $n-1$. We reformulate Theorem 6.5 in these terms.

**Theorem 7.1.** *Let $D$ be a strong local tournament on $n$ vertices with at least $n+2$ arcs. Then $D$ has exactly two cycles of length $n-1$ if and only if $D$ is isomorphic to $Q_n$, $Q_5^*$ or $C_n^j$ for an integer $1 \leq j \leq n$.*

In this chapter we further investigate the following problem.

**Problem 7.2.** *Given a strong local tournament $D$ on $n$ vertices and an integer $r$ with $3 \leq r \leq n$. How many cycles of length $r$ exist in $D$?*

In this context Moon [57] proved the following result for tournaments in 1966.

**Theorem 7.3** (Moon [57] 1966)**.** *Let $T$ be a strong tournament on $n$ vertices and let $r$ be an integer such that $3 \leq r \leq n$. Then $T$ has at least $n-r+1$ cycles of length $r$ for every $3 \leq r \leq n$.*

In 1975, Las Vergnas [51] characterized all strongly connected tournaments with minimal number of cycles of a given length $4 \leq r \leq n-1$.

**Theorem 7.4** (Las Vergnas [51] 1975)**.** *Let $T$ be a strong tournament on $n \geq 5$ vertices. Then $T$ has at least $n-r+2$ cycles of length $r$ for every $4 \leq r \leq n-1$ unless $T$ is isomorphic to $Q_n$.*

The class of strongly connected tournaments with exactly one Hamiltonian cycle was characterized by Douglas [29] in 1970 and the class of strongly connected tournaments with exactly $n-2$ cycles of length three was characterized by Burzio and Demaria [24] in 1990.

Bang-Jensen, Guo, Gutin and Volkmann [8] investigated which local tournaments are vertex pancyclic and proved the following result.

**Theorem 7.5** (Bang-Jensen, Guo, Gutin & Volkmann [8] 1997). *Let $D$ be a strongly connected local tournament on $n \geq 4$ vertices that is not a cycle.*

*(a) If $D$ is round-decomposable, then $D$ is vertex $(g(D)+1)$-pancyclic, where $g(D)$ is the length of a shortest cycle in the round decomposition of $D$;*

*(b) If $D$ is not round-decomposable, then $D$ is vertex pancyclic.*

In this chapter we investigate Problem 7.2 and transfer Theorems 7.3 and 7.4 to the class of local tournaments.

## 7.1 Observations on the structure of local tournaments that are not round-decomposable

By refining the proof of Theorem 1.17 we can show that a strongly connected local tournament that is not a tournament has the following structure.

**Theorem 7.6.** *Let $D$ be a strong local tournament which is not a tournament and not round-decomposable. Let $S$ be a minimal separating set of $D$ that satisfies the conditions of Theorem 1.17. Then one of the following possibilities holds.*

*(a) There is a vertex $s \in S$ and vertices $x_1, x_2 \in V(D_2')$ with $x_1 \to s \to x_2$ and $D$ has a Hamiltonian cycle $C$ such that $x_1$ is the predecessor of $x_2$ on $C$;*

*(b) There is a vertex $x \in V(D_2')$ and vertices $s_1, s_2 \in S$ with $s_1 \to x \to s_2$ and $D$ has a Hamiltonian cycle $C$ such that $s_1$ is the predecessor of $s_2$ on $C$.*

*Proof.* Let $S$ be a minimal separating set of $D$ such that $D-S$ is not a tournament, $D[S]$ is a tournament and the semicomplete decomposition of $D-S$ has exactly three components $D_1'$, $D_2'$ and $D_3'$. Let $\alpha$, $\beta$, $\mu$ and $\nu$ be integers with $\lambda \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that

$$N^-(D_\alpha, D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha, D_\nu) \neq \emptyset$$
$$\text{or } N^-(D_\mu, D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu, D_\beta) \neq \emptyset,$$

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and $D_\lambda$ is the initial component of $D'_2$. By symmetry we may assume, without loss of generality, that $N^-(D_\alpha, D_\mu) \neq \emptyset$ and $N^+(D_\alpha, D_\nu) \neq \emptyset$. Let $\mu$ and $\nu$ be chosen such that $\nu - \mu$ is minimal. Furthermore, let $s_1 \in V(D_\mu)$, $s_2 \in V(D_\nu)$ and $\{x_1, x_2\} \subseteq V(D_\alpha)$ be vertices of $D$ such that $s_1 \to x_1$ and $x_2 \to s_2$ (it is possible that $s_1 = s_2$ or $x_1 = x_2$).

If $s_1 \not\to D_\alpha$ ($D_\alpha \not\to s_2$), there exists a vertex $x \in V(D_\alpha)$ such that $x \to s_1 \to x^+$ ($x \to s_2 \to x^+$), i.e. (a) holds. So assume that $s_1 \to D_\alpha \to s_2$.

If $D_\alpha \not\to D_\nu$, there exists a vertex $x \in V(D_\alpha)$ and a vertex $s \in V(D_\nu)$ such that $s \to x \to s^+$, i.e. (b) holds. So assume that $D_\alpha \to D_\nu$.

Analogously, if $D_\mu \not\to D_\alpha$, there exists a vertex $x \in V(D_\alpha)$ and a vertex $s \in V(D_\mu)$ such that $x \to s \to x^+$, i.e. (a) holds. So assume that $D_\mu \to D_\alpha$.

If $\mu + 1 = \nu$, note that $s_1 \to x_i \to s_2$ for $i = 1, 2$, i.e. (b) holds. So assume that $\mu + 1 < \nu$. But then $N^-(D_{\mu+1}, D_\alpha) \neq \emptyset$ or $N^+(D_{\mu+1}, D_\alpha) \neq \emptyset$, both contradictions to the choice of $\mu$ and $\nu$. □

Using the structure obtained in Theorem 7.6, we can show the following lemma which is a preparatory result for Theorem 7.9.

**Lemma 7.7.** *Let $D$ be a strong local tournament on $n$ vertices which is not a tournament. If $D$ is not round-decomposable, then $D$ has $n - 5$ distinct vertices $v_1, v_2, \ldots, v_{n-5}$ such that for every $i = 1, 2, \ldots, n-5$ the digraph $D - v_i$ is a strong, not round-decomposable local tournament which is not a tournament.*

*Proof.* Let $S$ be a minimal separating set of $D$ that satisfies the conditions of Theorem 1.17. By Theorem 7.6 there exist integers $\alpha$, $\beta$, $\mu$ and $\nu$ with $\lambda \leq \alpha \leq \beta \leq p - 1$ and $p + 1 \leq \mu \leq \nu \leq p + q$ such that either

(a) there is a vertex $s \in S$ and vertices $x_1, x_2 \in V(D'_2)$ with $x_1 \to s \to x_2$ and $D$ has a Hamiltonian cycle $C$ such that $x_1$ is the predecessor of $x_2$ on $C$ or

(b) there is a vertex $x \in V(D'_2)$ and vertices $s_1, s_2 \in S$ with $s_1 \to x \to s_2$ and $D$ has a Hamiltonian cycle $C$ such that $s_1$ is the predecessor of $s_2$ on $C$,

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and $D_\lambda$ is the initial component of $D'_2$. By symmetry we may assume, without loss of generality, that (a) holds.

In this case for every vertex $v \in (V(D'_2) \cup S) - \{s, x_1, x_2\}$ the digraph $D - v$ is a strong local tournament which is not a tournament and not round-decomposable.

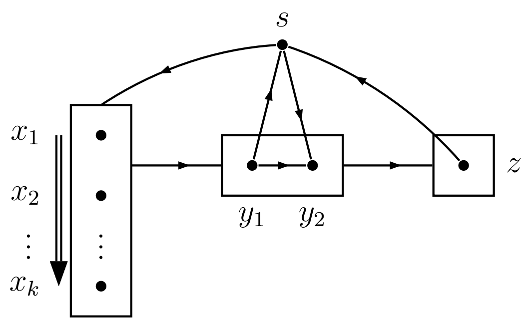


Figure 7.1: A class of local tournaments with $n-3$ cycles of length three.

In addition, if $|V(D'_j)| \geq 3$ for an index $j \in \{1,3\}$, then for every vertex $v \in V(D'_j)$ the digraph $D-v$ is a strong local tournament which is not a tournament and not round-decomposable. □

## 7.2 A generalization of Las Vergnas' Theorem

In this section we transfer Theorem 7.3 to the class of local tournaments. Firstly we consider strongly connected local tournaments that are not round-decomposable. The next example shows that Theorem 7.9 is not correct for $r = 3$ if the local tournament in question is not a tournament (cf. Figure 7.1).

**Example 7.8.** *Let $k \geq 1$ be an integer and let $D$ be the local tournament with vertex set*

$$V = \{s\} \cup \{x_i \mid i = 1, 2, \ldots, k\} \cup \{y_1, y_2\} \cup \{z\}$$

*and arc set*

$$\begin{aligned} E = & \{x_i x_j \mid 1 \leq i < j \leq k\} \cup \{y_1 y_2\} \cup \{s x_i \mid i = 1, 2, \ldots, k\} \\ & \cup \{x_i y_j \mid i = 1, 2, \ldots, k, \ j = 1, 2\} \cup \{y_i z \mid i = 1, 2\} \\ & \cup \{zs\} \cup \{y_1 s\} \cup \{s y_2\}. \end{aligned}$$

*Then $|V| = k+4$ and the number of cycles of length 3 in $D$ is $k+1 = |V|-3$.*

However we can show the following result.

**Theorem 7.9.** *Let $D$ be a strong local tournament on $n \geq 5$ vertices that is not round decomposable. Then $D$ has at least $n-r+1$ cycles of length $r$ for every $4 \leq r \leq n$.*

*Proof.* We shall prove the proposition by induction on $n$.

**Induction basis.** Let $n = 5$. Since $D$ is strong, it has a Hamiltonian cycle by Theorem 1.13. Furthermore, since $D$ is not round-decomposable, it has at least $n + 2$ arcs. Then $D$ has at least two cycles of length four by Theorem 6.1.

**Inductive step.** Let $n \geq 6$ be an integer. If $D$ is not a tournament, it has a vertex $x$ such that $D - x$ is a strong local tournament that is not round-decomposable by Lemma 7.7. If $D$ is a tournament, it has a non-separating vertex $x$ by Theorem 7.3. By the induction hypothesis, the digraph $D - x$ has at least $(n-1) - r + 1$ cycles of length $r$. Since $D$ is either a tournament or a local tournament that is not round-decomposable and not a tournament, it is vertex pancyclic by Theorem 7.3 or 7.5. It follows that there is a cycle of length $r$ through $x$ in $D$. Therefore $D$ has at least $n - r + 1$ cycles of length $r$. □

In the next theorem we characterize all local tournaments for which the inequality in Theorem 7.9 is sharp.

**Theorem 7.10.** *Let $D$ be a strong local tournament on $n \geq 5$ vertices that is not round-decomposable. If $D$ is neither isomorphic to $Q_n$ nor to $Q_5^*$, then $D$ has at least $n - r + 2$ cycles of length $r$ for every $4 \leq r \leq n - 1$.*

*Proof.* Note that $D$ is vertex pancyclic by Theorem 7.5. We prove the proposition by induction on $n$.

**Induction basis.** Let $n = 5$. We have to show that $D$ has at least three cycles of length four, in other words, that $D$ has at least three non-separating vertices. Note that $D$ has at least $n + 2 = 7$ arcs, since it is not round-decomposable. Furthermore, $D$ is not isomorphic to $C_5^j$ for $j = 1, 2, \ldots, 5$. Moreover, $D$ is neither isomorphic to $Q_5$ nor to $Q_5^*$. It follows by Theorem 6.5 that $D$ has at least three non-separating vertices.

**Inductive step.** Let $n \geq 6$. If $D$ is not a tournament, it has a vertex $x$ such that $D - x$ is a strong local tournament that is not round-decomposable by Lemma 7.7. If $D$ is a tournament, it has a non-separating vertex $x$ by Theorem 7.3. We consider three cases.

*Case 1:* Suppose that $D - x$ is neither isomorphic to $Q_{n-1}$ nor to $Q_5^*$. Then, by the induction hypothesis, $D - x$ has $(n-1) - r + 2 = n - r + 1$ cycles of length $r$. Since $D$ has a cycle of length $r$ through $x$, the proof is complete in this case.

*Case 2:* Suppose that $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and $D$ has two cycles of length $r$ through $x$. Recall that $Q_{n-1}$ and $Q_5^*$ have exactly $n - r$ cycles of length $r$. Then $D$ has $n - r + 2$ cycles of length $r$ and the proof of this case is complete.

*Case 3:* Suppose that $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and $D$ has exactly one cycle of length $r$ through $x$. We consider two subcases.

*Case 3.1:* Suppose that $n \geq 8$. If $D$ is a tournament, it has three vertices $u, v, w$ such that $D - u$, $D - v$ and $D - w$ are strong tournaments by Theorem 7.4. If $D$ is not a tournament, it has three vertices $u, v, w$ such that $D - u$, $D - v$ and $D - w$ are strong, not round-decomposable local tournaments that are not tournaments by Lemma 7.7.

We assume that $D - x \cong Q_{n-1}$ for every vertex $x \in \{u, v, w\}$ and that there is exactly one cycle of length $r$ in $D$ through each of the vertices $u, v, w$.

Let $\{y_1, y_2, \ldots, y_{n-1}\}$ be the vertex set of $D - u$ such that $y_i \rightarrow y_{i+1}$ and $y_k \rightarrow y_j$ and let $\{z_1, z_2, \ldots, z_{n-1}\}$ be the vertex set of $D - v$ such that $z_i \rightarrow z_{i+1}$ and $z_k \rightarrow z_j$ for $1 \leq i \leq n-2$ and $1 \leq j < k \leq n-1$ with $j \neq k+1$. Note that the only vertices that belong to exactly one cycle of length $r$ in $D - u$ and $D - v$ are $\{y_1, y_{n-1}\}$ and $\{z_1, z_{n-1}\}$, respectively. It follows that $\{v, w\} = \{y_1, y_{n-1}\}$ and $\{u, w\} = \{z_1, z_{n-1}\}$. Let, without loss of generality, $y_1 = v$ and $y_{n-1} = w$. Since $d^-(w, D - u) = 1$, it follows that $d^-(w, D - v) \leq 2$. In addition, since $d^-(z_1, D - v) = n - 3 > 2$, it follows that $z_1 = u$ and $z_{n-1} = w$. Since $d^+(y_2, D - u) = 1$, it follows that $d^-(y_2, D - v) \leq 2$. The latter implies that $y_2 \in \{z_2, z_3\}$.

*Case 3.1.1:* Suppose that $y_2 = z_3$. Note that $v$ is dominated by $V(D) - \{u, y_2\}$ and dominates $y_2$. It follows that $vz_3z_4 \ldots z_rz_2v$ and $vz_3z_4 \ldots z_{r-1}z_1z_2v$ are two cycles of length $r$ through $v$, a contradiction.

*Case 3.1.2:* Suppose that $y_2 = z_2$. Note again that $v$ is dominated by $V(D) - \{u, y_2\}$ and dominates $y_2$. Since $D$ is a local tournament, it follows that $z_1 = u$ and $v$ are adjacent. If $v$ is dominated by $u$, the cycles $vz_2z_3 \ldots z_{r-1}z_1v$ and $vz_2z_3 \ldots z_rv$ are two cycles of length $r$ through $v$, a contradiction. So assume that $v$ dominates $u$. But then $vz_1z_2 \ldots z_{r-1}v$ and $vz_2z_3 \ldots z_rv$ are two cycles of length $r$ through $v$, again a contradiction.

*Case 3.2:* Suppose that $n \in \{6, 7\}$. We assume that $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and that there is exactly one cycle of length $r$ in $D$ through $x$. Let $\{y_1, y_2, \ldots, y_{n-1}\}$ be the vertex set of $D - x$ such that $y_i \rightarrow y_{i+1}$ and $y_k \rightarrow y_j$ for $1 \leq i \leq n-2$ and $1 \leq j < k \leq n-1$ with $j \neq k+1$ (if $D - x \cong Q_5^*$, there is no arc between $y_2$ and $y_4$). Since $D$ is strong, the vertex $x$ has at least one out- and one in-neighbor in $D - x$. It follows that there exists an index $1 \leq i \leq n$ such that $y_i \rightarrow x \rightarrow y_{i+1}$. By Theorem 6.5 we may assume that $r \leq n-2$. Hence we consider three remaining cases.

*Case 3.2.1:* Suppose that $r = 4$ and $n = 6$. We show that there exist two cycles of length four through $x$ without using the arc $y_4y_2$. Note that the case $i = 1$ and $i = 4$ as well as the cases $i = 2$ and $i = 3$ are symmetrical.

If $i = 1$, the cycle $xy_2y_3y_1x$ is a 4-cycle through $x$. In addition, since $\{x, y_5\} \to y_2$, the vertices $x$ and $y_5$ are adjacent. If $x \to y_5$, the cycle $xy_5y_3y_1x$ is a second 4-cycle through $x$, a contradiction. So assume that $y_5 \to x$. Then, since $y_5 \to \{x, y_3\}$, the vertices $x$ and $y_3$ are adjacent. If $x \to y_3$, the cycle $xy_3y_4y_5x$ is a second 4-cycle through $x$, a contradiction. So assume that $y_3 \to x$. Since $y_3 \to \{x, y_4\}$, the vertices $x$ and $y_4$ are adjacent. If $y_4 \to x$, the cycle $xy_2y_3y_4x$ is a second 4-cycle through $x$ and if $x \to y_4$, the cycle $xy_4y_5y_1x$ is a second 4-cycle through $x$, both possibilities are contradictions.

If $i = 2$, the cycle $xy_3y_1y_2x$ is a 4-cycle through $x$. In addition, since $\{x, y_5\} \to y_3$, the vertices $x$ and $y_5$ are adjacent. If $x \to y_5$, the cycle $xy_5y_1y_2x$ is a second 4-cycle through $x$ and if $y_5 \to x$, the cycle $xy_3y_4y_5x$ is a second 4-cycle through $x$, both possibilities are contradictions.

If $i = 5$, there is an arc between $x$ and each of the vertices $y_j$ for $j = 2, 3, 4$, since $\{x, y_3, y_4\} \to y_1$ and $y_5 \to \{x, y_2\}$. If $x \to \{y_2, y_3, y_4\}$ or if $\{y_2, y_3, y_4\} \to x$, the local tournament $D$ is isomorphic to $Q_6$, a contradiction. If $y_4 \to x \to \{y_2, y_3\}$ or if $\{y_3, y_4\} \to x \to y_2$, the cycle $xy_2y_3y_4x$ is a 4-cycle through $x$. A second cycle of length four is given by $xy_3y_4y_5x$ and $xy_1y_2y_3x$, respectively, again a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.1 is complete.

*Case 3.2.2:* Suppose that $r = 4$ and $n = 7$. Note that the cases $i = 1$ and $i = 5$ are symmetrical.

If $2 \le i \le 4$, the cycles $xy_{i+1}y_{i-1}y_ix$ and $xy_{i+1}y_{i+2}y_ix$ are two cycles of length four through $x$, a contradiction.

If $i = 1$, the cycle $xy_2y_3y_1x$ is a 4-cycle through $x$. In addition, since $\{x, y_4\} \to y_2$, the vertices $x$ and $y_4$ are adjacent. If $x \to y_4$, the cycle $xy_4y_5y_1x$ is a second 4-cycle through $x$ and if $y_4 \to x$, the cycle $xy_2y_3y_4x$ is a second 4-cycle through $x$, both possibilities are contradictions.

If $i = 6$, there is an arc between $x$ and each of the vertices $y_j$ for $j = 2, 3, 4, 5$, since $\{x, y_4, y_5\} \to y_1$ and $y_6 \to \{x, y_2, y_3\}$. If $x \to \{y_2, y_3, y_4, y_5\}$ or if $\{y_2, y_3, y_4, y_5\} \to x$, the digraph $D$ is isomorphic to $Q_7$, a contradiction. If $y_5 \to x \to \{y_2, y_3, y_4\}$, the cycles $xy_3y_4y_5x$ and $xy_4y_5y_6x$ are two cycles of length four through $x$; if $\{y_3, y_4, y_5\} \to x \to y_2$, the cycles $xy_2y_3y_4x$ and $xy_1y_2y_3x$ are two cycles of length four through $x$; and if $\{y_4, y_5\} \to x \to \{y_2, y_3\}$, the cycles $xy_2y_3y_4x$ and $xy_3y_4y_5x$ are two cycles of length four through $x$. Thus, all three possibilities yield a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.2 is complete.

*Case 3.2.3:* Suppose that $r = 5$ and $n = 7$. Note that the cases $i = 1$ and $i = 5$ are symmetrical.

If $i = 1$, the cycle $xy_2y_3y_4y_1x$ is a 5-cycle through $x$. In addition, since $\{x, y_5\} \rightarrow y_2$, the vertices $x$ and $y_5$ are adjacent. If $y_5 \rightarrow x$, the cycle $xy_2y_3y_4y_5x$ is a second 5-cycle through $x$ and if $x \rightarrow y_5$, the cycle $xy_5y_3y_4y_1x$ is a second 5-cycle through $x$. Both possibilities yield a contradiction.

If $i = 2$, the cycles $xy_3y_4y_1y_2x$ and $xy_3y_4y_5y_2x$ are two cycles of length five through $x$, a contradiction.

If $i \in \{3, 4\}$, the cycles $xy_{i+1}y_{i-2}y_{i-1}y_ix$ and $xy_{i+1}y_{i+2}y_{i-1}y_ix$ are two cycles of length five through $x$, a contradiction.

If $i = 6$, there is an arc between $x$ and each of the vertices $y_j$ for $j = 2, 3, 4, 5$, since $\{x, y_4, y_5\} \rightarrow y_1$ and $y_6 \rightarrow \{x, y_2, y_3\}$. If $x \rightarrow \{y_2, y_3, y_4, y_5\}$ or if $\{y_2, y_3, y_4, y_5\} \rightarrow x$, the local tournament $D$ is isomorphic to $Q_7$, a contradiction. If $y_5 \rightarrow x \rightarrow \{y_2, y_3, y_4\}$ or if $\{y_4, y_5\} \rightarrow x \rightarrow \{y_2, y_3\}$, the cycles $xy_2y_3y_4y_5x$ and $xy_3y_4y_5y_6x$ are two cycles of length five through $x$, a contradiction. If $\{y_3, y_4, y_5\} \rightarrow x \rightarrow y_2$, the cycles $xy_2y_3y_4y_5x$ and $xy_1y_2y_3y_4x$ are two cycles of length five through $x$, again a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.3 and therefore the proof of this theorem is complete. □

We make the following observation.

**Remark 7.11.** *Let $D$ be a strong local tournament that is not round-decomposable and not a tournament. If $D$ does not fulfill the conclusion of Theorem 7.10, then $D$ is isomorphic to $Q_5^*$.*

Now we consider round-decomposable local tournaments to complete our investigation.

**Theorem 7.12.** *Let $D$ be a strongly connected and round-decomposable local tournament with the round decomposition $R[D_1, D_2, \ldots, D_p]$, where $p \geq 3$. Let $C_R$ be a shortest cycle of length $g(D)$ in $R$. Then*

*(a) $D$ has at least $n - r + 2$ cycles of length $r$ for every $g(D) + 2 \leq r \leq n - 1$;*

*(b) $D$ has at least $n - g(D)$ cycles of length $g(D) + 1$ with equality if and only if*

- *(i) $|V(D_i)| = 1$ for every component $D_i \in V(C_R)$;*
- *(ii) if $D_i \in V(C_R)$, then $D - x$ is not strong for every vertex $x \in V(D_i)$;*
- *(iii) if $D_j \notin V(C_R)$, then $|V(D_j)| \leq g(D)$.*

*Proof.* Let

$$C_R = D_1 D_{\alpha_2} D_{\alpha_3} \dots D_{\alpha_g} D_1$$

be a cycle of length $g = g(D)$ in $R$, where $1 < \alpha_2 < \alpha_3 < \dots < \alpha_g$. The cycle $C_R$ induces a cycle

$$C = x_1 x_{\alpha_2} x_{\alpha_3} \dots x_{\alpha_g} x_1$$

in $D$, where $x_i \in V(D_i)$. Note that every vertex $v \notin V(C)$ has a positive as well as a negative neighbor on $C$. Let $S$ be an arbitrary set of vertices such that $v \notin V(C)$ for every vertex $v \in S$. Then all vertices of $S$ can be inserted in $C$ to construct a cycle of length $g(D) + |S|$ in $D$ by Lemma 1.12. It follows that there are at least $\binom{n-g(D)}{s}$ cycles of length $g(D) + s$ in $D$, where $0 \leq s \leq n - g(D)$.

If $2 \leq s \leq n - g(D) - 1$, we derive $g(D) + 2 \leq g(D) + s = r \leq n - 1$ and

$$\binom{n - g(D)}{s} \geq n - g(D) \geq n - r + 2,$$

since $g(D) + 2 \leq r$. So (a) is true.

If $s = 1$, we deduce that $D$ has at least

$$\binom{n - g(D)}{1} = n - g(D)$$

cycles of length $g(D) + s = g(D) + 1$. Note that all these cycles include at least one vertex of every component $D_i \in V(C_R)$. To prove (b) assume that one of the conditions (i), (ii) or (iii) is not satisfied.

*Case 1.* Suppose that $|V(D_i)| \geq 3$ for a component $D_i$ of $C_R$. Then the cycle $C_R$ induces at least $|V(D_i)|$ cycles of length $g(D)$ in $D$. Since $|V(D_i)| - 1 \leq n - g(D)$ and $|V(D_i)| \geq 3$, it follows that there are at least

$$\begin{aligned}
|V(D_i)|(n - g(D) + 1 - |V(D_i)|) &+ \binom{|V(D_i)|}{2} \\
&= n - g(D) + 1 + (|V(D_i)| - 1)(n - g(D) + 1) - \frac{|V(D_i)|^2 + |V(D_i)|}{2} \\
&\geq n - g(D) + 1 + (|V(D_i)| - 1)|V(D_i)| - \frac{|V(D_i)|^2 + |V(D_i)|}{2} \\
&= n - g(D) + 1 + \frac{|V(D_i)|(|V(D_i)| - 3)}{2} \\
&\geq n - g(D) + 1
\end{aligned}$$

cycles of length $g(D) + 1$ in $D$.

*Case 2.* Suppose that $D - x$ is strong for a vertex $x \in V(D_i)$, where $D_i$ is a component of $C_R$. Then there exists a cycle of length $g(D)+1$ in $D$ that includes no vertex of $D_i$. Therefore there are at least $n-g(D)+1$ cycles of length $g(D)+1$ in $D$.

*Case 3.* Suppose that $|V(D_j)| \geq g(D)+1$ for a component $D_j$ that is not on $C_R$. Since $D_j$ induces a strong subtournament of $D$, the component $D_j$ is pancyclic by Theorem 7.3. In particular there exists a cycle of length $g(D)+1$ in $D_j$ and this cycle includes no vertex of $D_i$ for every component $D_i$ of $C_R$. Therefore there are at least $n - g(D) + 1$ cycles of length $g(D) + 1$ in $D$.

Assume now that the conditions (i)-(iii) are satisfied. In this case the cycle $C_R$ induces a unique cycle $C$ of length $g(D)$ in $D$. Due to the conditions (i)-(iii) there does not exist another cycle of length $g(D)$ in $D$. Therefore a cycle of length $g(D) + 1$ in $D$ consists of all vertices of $C$ and another (arbitrary) vertex of $D$. It follows that $D$ has exactly $n - g(D)$ cycles of length $g(D) + 1$ and the proof of this theorem is complete. □

As a corollary of Theorem 7.10 and Theorem 7.12 we derive Theorem 7.4.

**Corollary 7.13** (Las Vergnas [51] 1975)**.** *Let $T$ be a strong tournament on $n \geq 5$ vertices. Then $T$ has at least $n - r + 2$ cycles of length $r$ for every $4 \leq r \leq n-1$ unless $T$ is isomorphic to $Q_n$.*

*Proof.* If $T$ is not round-decomposable, the conclusion is valid by Theorem 7.10. So assume that $D$ is a round-decomposable tournament with the round decomposition $R[D_1, D_2, \ldots, D_p]$. Then $g(D) = 3$ and thus, $D$ has $n - r + 2$ cycles of length $r$ for $5 \leq r \leq n-1$ in view of Theorem 7.12.(a). Assume now that $D$ has only $n - 3$ cycles of length four. Since every vertex of $R$ is on a 3-cycle, we conclude that $|V(D_i)| = 1$ for every component $D_i$ by Theorem 7.12.(b).(i). This means that $D = R$. But now Theorem 7.12.(b).(ii) implies that every vertex of $D$ is a separating vertex, a contradiction to Corollary 5.1. □

# Chapter 8

# The partition of a strong local tournament

In 1985, Reid [61] showed that every 2-connected tournament $T$ on $n \geq 6$ vertices can be partitioned into two complementary cycles of lengths 3 and $n-3$ unless $T$ is isomorphic to $T_7^1$, where $T_7^1$ is the tournament on seven vertices that contains no transitive subtournament on four vertices (cf. Figure 8.1).

**Theorem 8.1** (Reid [61] 1985)**.** *Let $T$ be a 2-connected tournament on $n \geq 6$ vertices. Then $T$ contains two vertex disjoint cycles of lengths 3 and $n-3$, unless $T$ is isomorphic to $T_7^1$.*

Using Reid's result as the induction basis, Song [64] completed this result in 1993.

**Theorem 8.2** (Song [64] 1993)**.** *Let $T$ be a 2-connected tournament on $n \geq 6$ vertices. Then $T$ contains two vertex disjoint cycles of lengths $s$ and $n-s$ for every $s$ with $3 \leq s \leq \frac{n}{2}$, unless $T$ is isomorphic to $T_7^1$.*

Recall that the class $\mathcal{R}_n^2$ consists of all round local tournaments that are 2-regular (cf. Definition 1.5). In 1994, Volkmann and Guo [37] characterized all 2-connected local tournaments that are not cycle complementary (cf. Figure 8.1).

**Theorem 8.3** (Guo & Volkmann [37] 1994)**.** *Let $D$ be a 2-connected locally semicomplete digraph on $n \geq 6$ vertices. Then $D$ contains two vertex disjoint cycles, unless $D$ is isomorphic to a member of $\{T_6^1, T_6^2, T_6^3, T_7^1, T_7^2\} \cup \mathcal{R}_n^2$.*

Generalizing their own result Guo and Volkmann showed that the length of the complementary cycles can be chosen somewhat arbitrary.

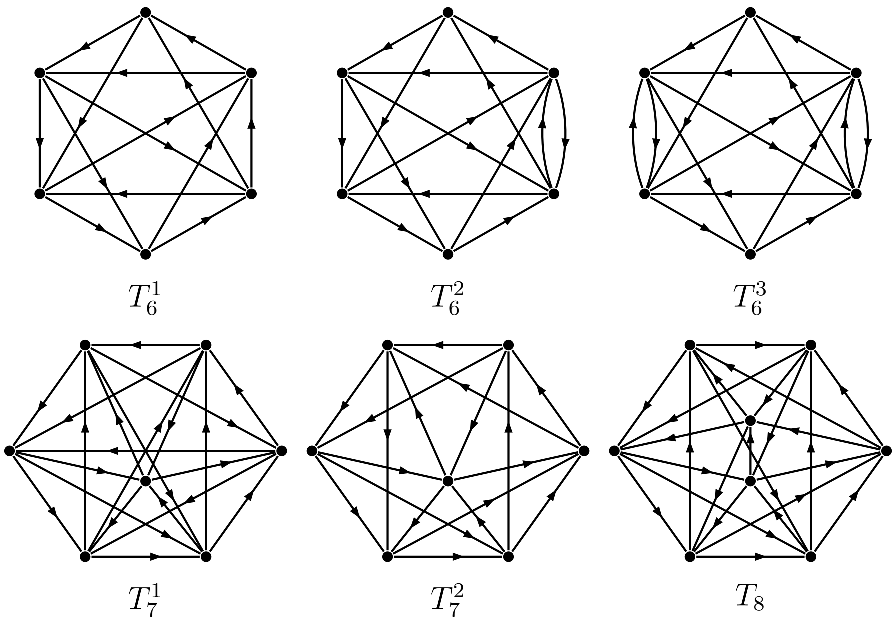


Figure 8.1: The locally semicomplete digraphs that are exceptions for Theorems 8.1, 8.2, 8.3 and 8.4.

**Theorem 8.4** (Guo & Volkmann [38] 1996)**.** *Let $D$ be a 2-connected locally semicomplete digraph on $n \geq 6$. Then $D$ contains two vertex disjoint cycles of lengths $s$ and $n - s$ for every $s$ with $g(D) \leq s \leq n - g(D)$, unless $D$ is isomorphic to a member of $\{T_6^1, T_6^2, T_6^3, T_7^1, T_7^2, T_8\} \cup \mathcal{R}_n^2$.*

Since $g(D) = 3$ if $D$ is a tournament, the above theorem is a generalization of Theorem 8.1.

In 2005, Li and Shu [53] studied the structure of strongly connected tournaments that cannot be partitioned into two cycles.

**Theorem 8.5** (Li & Shu [53] 2005)**.** *Let $T$ be a strong tournament on $n \geq 6$ vertices that is not isomorphic to $T_7^1$ and let $c$ be the number of separating vertices of $T$. If $T$ is not cycle complementary, then*

*(a)* $\delta^+(T) \leq 2$ *and* $\delta^-(T) \leq 2$*;*

*(b) there* exists *a Hamiltonian cycle $C$ of $T$ that can be partitioned into two consecutive segments $Q$ and $L$ such that $Q$ induces a transitive tournament, $L$ induces a tournament that is isomorphic to $Q_{|L|}$, all separating vertices of $T$ are consecutive on $L$ and $|L| = c, c+1$ or $c+2$.*

In particular, they showed that a strong tournament $T$ with $n \geq 6$ vertices that is not isomorphic to $T_7^1$ and has minimal degree at least three is cycle complementary.

**Corollary 8.6** (Li & Shu [53] 2005)**.** *Let $T$ be a strong tournament on $n \geq 6$ vertices such that $T$ is not isomorphic to $T_7^1$. If* $\max\{\delta^-(T), \delta^+(T)\} \geq 3$*, then $T$ is cycle complementary.*

Using Corollary 8.6 as the induction basis, Li and Shu [52] proved a sufficient condition for a strong tournament to have a $k$-cycle factor.

**Theorem 8.7** (Li & Shu [52] 2004)**.** *Let $T$ be a strong tournament on $n$ vertices that is not isomorphic to $T_7^1$ and let $k \geq 2$ be a positive integer. If*

$$\delta^+(T) + \delta^-(T) \geq \frac{k-2}{k-1}n + 3k - 1,$$

*then $T$ has a $k$-cycle factor.*

This problem - whether a strongly connected tournament can be partitioned into $k$ cycles - was treated by Chen, Gould and Li [28] in 2001. They showed that every $k$-connected tournament with at least $8k$ vertices has a $k$-cycle factor.

**Theorem 8.8** (Chen, Gould & Li [28] 2001)**.** *Let $T$ be a $k$-connected tournament on $n \geq 8k$ vertices. Then $T$ has a $k$-cycle factor.*

In 2004, Gould and Guo [32] discussed the same problem for the class of locally semicomplete digraphs. Generalizing the work of Chen, Gould and Li, the authors showed the following results.

**Theorem 8.9** (Gould & Guo [32] 2004)**.** *Let $D$ be a $k$-connected locally semicomplete digraph on $n \geq 5k + 1$ vertices. Then $D$ has a $k$-cycle factor such that one of the cycle has length at most five.*

**Theorem 8.10** (Gould & Guo [32] 2004)**.** *Let $D$ be a $k$-connected locally semicomplete digraph that is round-decomposable. If $D$ contains $k + 1$ vertex disjoint cycles of length three, then $D$ has a $k$-cycle factor such that at least $k - 2$ of the cycles have length three.*

Inspired by the articles of Li and Shu [52, 53] we develop in this chapter the structure necessary for a strongly connected local tournament to be not cycle complementary. Using this structure, we generalize Li's and Shu's conditions to local tournaments. Firstly we would like to make the following remark.

**Remark 8.11.** *Li and Shu stated that in the situation of Theorem 8.5.(b) the Hamiltonian cycle $C$ can be chosen* arbitrary*. This is not true as the tournament depicted in Figure 8.2 shows. The vertices $u$, $v$ and $w$ are the only separating vertices of $T$, but $T$ has a Hamiltonian cycle $C$ that starts with $ux$ and ends with $yu$. So $u$, $v$ and $w$ are not consecutive on $C$.*

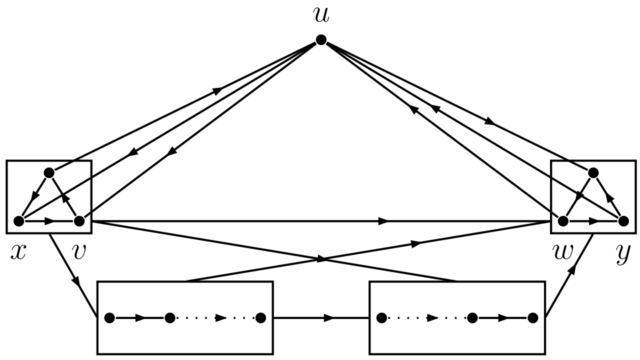


Figure 8.2: A counterexample for Theorem 8.5.(b).

## 8.1 Structure of local tournaments that are not cycle complementary

In this section we develop the structure necessary for a strongly connected local tournament to be not cycle complementary. We start with a first lemma.

**Lemma 8.12.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$.*

*(a) $|V(D_i)| \leq 3$ for every index $i$ with $2 \leq i \leq p-1$;*

*(b) If $|V(D_i)| = 3$ for an index $i$ with $2 \leq i \leq p-1$, then there is no arc from $D_r$ to $D_s$ for every pair $r, s$ of indices with $r < i < s$.*

*(c) If $u \to D_1$, then $|V(D_1)| \leq 3$ and if $D_p \to u$, then $|V(D_p)| \leq 3$;*

*(d) If $|V(D_1)| \geq 3$, then $u$ has no out-neighbor in $D_2$ and if $|V(D_p)| \geq 3$, then $u$ has no in-neighbor in $D_{p-1}$;*

*Proof.* Assume that $|V(D_i)| \geq 4$ for an index $i$ with $2 \leq i \leq p-1$. Note that $D_i$ is a strong subtournament of $D$. According to Theorem 2.3, the component $D_i$ is pancyclic and thus, contains a cycle $C_i$ of length $|V(D_i)| - 1$. Let $x_i \in V(D_i)$ be the vertex that does not belong to $C_i$. Then

$$uD_1D_2 \ldots D_{i-1}C_iD_{i+1}D_{i+2} \ldots D_pu$$

and $C_i$ are complementary cycles of $D$, a contradiction. So $|V(D_i)| \leq 3$ for every index $i$ with $2 \leq i \leq p-1$ and (a) has been proved.

To prove (b) let $|V(D_i)| = 3$ for an index $i$ with $2 \leq i \leq p-1$ and assume that there is an arc from $D_r$ to $D_s$ for a pair $r, s$ of indices with $r < i < s$. In view of Theorem 1.6 we obtain $D_{i-1} \to D_{i+1}$ and thus, $D_i$ and $D - D_i$ are strong complementary subdigraphs of $D$, a contradiction.

Analogously to the proof of (a) we can show that if $u \to D_1$, then $|V(D_1)| \leq 3$ and if $D_p \to u$, then $|V(D_p)| \leq 3$. So (c) is valid.

If $|V(D_1)| \geq 3$ and $N^+(u, D_2) \neq \emptyset$ (or if $|V(D_p)| \geq 3$ and $N^-(u, D_{p-1}) \neq \emptyset$), it is easy to see that $D_1$ and $D - D_1$ ($D_p$ and $D - D_p$, respectively) are strong complementary subdigraphs of $D$, a contradiction. Hence (d) is true and the proof is complete. □

So if $D_p \to u \to D_1$, the structure of $D$ is clear. The next two lemmas cover the remaining cases.

**Lemma 8.13.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $D_p \not\to u$, then $D_p - x$ is not strong for every in-neighbor $x$ of $u$ in $D_p$. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - x$, where $q \geq 2$. Then*

*(a)* $D_i \to D_j$ *for every pair of indices* $1 \leq i < j \leq p$;

*(b)* $|V(D_i)| = 1$ *for every index $i$ with* $2 \leq i \leq p-1$;

*(c)* $V(A_i) = \{a_i\}$ *for every index* $i < q$;

*(d)* $a_i \not\to u$ *for every index $i$ with* $2 \leq i \leq q-1$;

*(e) if* $x \to a_2$, *then* $a_1 \not\to u$;

*(f) if* $a_{q-1} \to x$, *then* $|V(A_q)| = 1$;

*(g) if* $a_i \to x$ *for an index* $i < q$, *then $u$ has no in-neighbors in* $A_q$;

*(h)* $|N^-(u, A_q)| \leq 1$ *and if* $|N^-(u, A_q)| = 1$, *then* $N^-(x, A_q) = N^-(u, A_q)$.

*Proof.* Since $u$ has in-neighbors both in $D_1$ and $D_p$, it follows by Theorem 1.6 that (a) is true and thus, (b) holds in view of Lemma 8.12.(b).

For the remaining part of this proof let $P$ be a Hamiltonian path of $D - (V(D_p) \cup \{u\})$ starting in $N^+(u, D_1)$ and ending in $D_{p-1}$. Let $x$ be an arbitrary in-neighbor of $u$ in $D_p$. If $D_p - x$ is strong, let $C_p$ be a Hamiltonian cycle of $D_p - x$. Then

$uPxu$ and $C_p$ are complementary cycles of $D$, a contradiction. Hence $D_p - x$ is not strong. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - x$, where $q \geq 2$.

If $|V(A_i)| \geq 3$ for an index $i < q$, a Hamiltonian cycle of $A_i$ and

$$uPA_1A_2 \ldots A_{i-1}A_{i+1}A_{i+2} \ldots A_qxu$$

are complementary in $D$, again a contradiction. So (c) is valid.

If $a_i \to u$ for an index $i$ with $2 \leq i \leq q-1$, the cycles

$$uPa_2a_3 \ldots a_iu \quad \text{and} \quad xa_1a_{i+1}a_{i+2} \ldots a_{q-1}A_qx$$

show that $D$ is cycle complementary, a contradiction, and hence (d) holds.

If $x \to a_2$ and $a_1 \to u$, we obtain a contradiction by the complementary cycles

$$uPa_1u \quad \text{and} \quad xa_2a_3 \ldots a_{q-1}A_qx.$$

Therefore (e) is true.

If $a_{q-1} \to x$ and $|V(A_q)| \geq 3$, let $C_q$ be a Hamiltonian cycle of $A_q$. Then $C_q$ and

$$uPa_1a_2 \ldots a_{q-1}xu$$

are complementary in $D$, a contradiction. So (f) has been proved.

If $a_i \to x$ for an index $i < q$ and $u$ has an in-neighbor in $A_q$, the cycles

$$uPa_{i+1}a_{i+2} \ldots a_{q-1}A_qu \quad \text{and} \quad xa_1a_2 \ldots a_ix$$

show that $D$ is cycle complementary, again a contradiction. Hence (g) holds.

Finally, let $v$ be an in-neighbor of $u$ in $A_q$. If $u$ has an in-neighbor $w \neq v$ in $A_q$, let $C_q$ be a Hamiltonian cycle of $A_q$. It follows that

$$uPC_q[v^+, w]u \quad \text{and} \quad xa_1a_2 \ldots a_{q-1}C_q[w^+, v]x$$

are complementary cycles in $D$. This final contradiction completes the proof of (h) and of this lemma. □

**Lemma 8.14.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $D_p \not\to u$, then $V(D_p)$ can be partitioned in $U^+ = N^+(u, D_p) \neq \emptyset$, $U^- = N^-(u, D_p) \neq \emptyset$ and $R = V(D_p) - U^+ - U^-$ such that*

*(a) $U^+ \to R \to U^-$;*

*(b) if $R = \emptyset$, then $|U^-| \leq 2$ and*

*(i) if $U^- = \{x_1, x_2\}$ such that $x_1 \to x_2$, then $x_1$ is the only in-neighbor of $x_2$ in $D_p$, the vertices are consecutive on every Hamiltonian cycle of $D_p$ and $x_1$ is a separating vertex of $D_p$ and $D$;*

*(ii) if $U^- = \{x_1\}$, then $x_1$ is a separating vertex of $D_p$ and $D$;*

*(c) if $R \neq \emptyset$, then $D[U^+]$ is transitive, $|U^-| = 1$ and $D[R]$ is transitive or a 3-cycle. Furthermore $D_1$ consists of a single vertex.*

*Proof.* We consider the following partition of $V(D_p)$. Let

$$\begin{aligned} U^+ &= \{x \in V(D_p) \mid u \to x\}, \\ U^- &= \{x \in V(D_p) \mid x \to u\} \text{ and} \\ R &= V(D_p) - U^+ - U^-. \end{aligned}$$

Note that $D_p$ induces a strong tournament in $D$. If there exists an arc leading from $U^-$ to $R$ or from $R$ to $U^+$, it follows that there exists an arc between $u$ and $R$, a contradiction to the definition of $R$. So $U^+ \to R \to U^-$ and (a) has been proved. Let $U^- = \{x_1, x_2, \ldots, x_r\}$, $U^+ = \{y_1, y_2, \ldots, y_s\}$ and $R = \{z_1, z_2, \ldots, z_t\}$, where $r, s \geq 1$ and $t \geq 0$.

For the remaining part of this proof let $P$ be a Hamiltonian path of $D - (V(D_p) \cup \{u\})$ starting in $N^+(u, D_1)$ and ending in $D_{p-1}$.

Note that $D_p - x_r$ has the structure as described in Lemma 8.13. In considering the cases $R = \emptyset$ and $R \neq \emptyset$ we shall show that (b) and (c) are valid.

*Case 1:* Suppose that $R = \emptyset$. Assume that $|U^-| \geq 3$. Note that $a_i \not\to u$ by Lemma 8.13.(d) and that $u$ has at most one in-neighbor in $A_q$ by Lemma 8.13.(h). Hence $a_1 \to u$ and $v \to u$ for a vertex $v \in V(A_q)$. Using Lemma 8.13.(g), it follows that $x \to a_2$, a contradiction to Lemma 8.13.(e). So $|U^-| \leq 2$. Let $C$ be a Hamiltonian cycle of $D_p$

If $U^- = \{x_1, x_2\}$ such that $x_1 \to x_2$ and $x_2$ has an in-neighbor $y \neq x_1$ in $D_p$, the cycles

$$uD_1D_2 \ldots D_{p-1}C[y^+, x_1]u \quad \text{and} \quad C[x_2, y]x_2$$

show that $D$ is cycle complementary, a contradiction. Therefore $x_1$ is the only in-neighbor of $x_2$ in $D_p$.

If $|U^-| = 1$ or if $|V(D_p)| = 3$, the vertices of $U^-$ are consecutive on $C$. So assume that $U^- = \{x_1, x_2\}$ such that $x_1 \to x_2$. Then $x_1$ is the only in-neighbor of $x_2$ in $D_p$ and hence the predecessor of $x_2$ on $C$. Hence the vertices of $U^-$ are consecutive on $C$.

If $U^- = \{x_1\}$ or if $U^- = \{x_1, x_2\}$ and $x_1 \to x_2$, we show that $x_1$ is a separating vertex of $D_p$ and $D$. Recall that $D_p - x_1$ is not strong by Lemma 8.13. If $U^- = \{x_1\}$, it is immediate that $D - x_1$ is not strong. So assume that $U^- = \{x_1, x_2\}$ such that $x_1 \to x_2$. Then $x_1$ is the only in-neighbor of $x_2$ in $D_p$ and thus, $x_1$ separates $x_2$ and $D_p - \{x_1, x_2\}$ in $D$.

All in all (b) is valid.

*Case 2.* Suppose that $R \neq \emptyset$. It follows that $u \to D_1$ and thus, $|V(D_1)| = 1$ or $|V(D_1)| = 3$ according to Lemma 8.12.(c). For the converse suppose that $|V(D_1)| = 3$. By Lemma 8.12.(d) the vertex $u$ has no outneighbor in $D_2$. Since all possible arcs between $u$ and $D_2$ exist in $D$, we conclude that $D_2 \to u$. But now $D_2 \to R$ and $D_2 \to u$ which implies the existence of an arc between $u$ and $R$, a contradiction to the definition of $R$. Therefore the component $D_1$ is just a single vertex.

If $D[U^+]$ contains a cycle $C$, let $Q_1$ be a Hamiltonian path of $D[U^+ - V(C)]$, let $Q_2$ be a Hamiltonian path of $D[R]$ and let $Q_3$ be a Hamiltonian path of $D[U^-]$. Then $C$ and $uPQ_1Q_2Q_3u$ are complementary cycles of $D$, a contradiction. So $U^+$ induces a transitive tournament in $D$. It follows that $D[U^+]$ has a unique Hamiltonian path, say $Q_1 = y_1y_2 \ldots y_s$.

If $|U^-| \geq 2$, let $Q_2$ be a Hamiltonian path of $D[R]$, let $x$ be an arbitrary in-neighbor of $y_1$ in $U^-$ and let $Q_3$ be a Hamiltonian path of $D[U^- - x]$. Then

$$y_1y_2 \ldots y_sQ_2xy_1 \quad \text{and} \quad uPQ_3u$$

are complementary cycles of $D$, a contradiction. Hence $|U^-| = 1$.

If $|R| \geq 4$ and $D[R]$ is not transitive, let $C$ be a cycle of length less than $|R|$ in $D[R]$ and let $Q_2$ be a Hamiltonian path of $D[R - V(C)]$. Then $C$ and

$$uPQ_1Q_2x_1u$$

are complementary cycles of $D$, a contradiction. So $|R| \leq 3$ or $D[R]$ is transitive which proves the last part of (c). □

Additionally we make the following observation.

**Lemma 8.15.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not 2-connected and not cycle complementary. If $D$ is not round-decomposable, then $D$ has a separating vertex $u$ such that the terminal strong component $D_p$ of the strong decomposition $D_1, D_2, \ldots, D_p$ of $D - u$, where $p \geq 2$, contains no separating vertex of $D$.*

*Proof.* Let $u$ be a separating vertex of $D$ such that the order of the terminal strong component $D_p$ of the strong decomposition $D_1, D_2, \ldots, D_p$, where $p \geq 2$, of $D - u$ is minimal. Assume that $D_p$ contains a separating vertex $v$ of $D$.

If $|V(D_p)| \geq 3$, let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - v$, where $q \geq 1$. Since $D - v$ is strong, the vertex $u$ has no in-neighbors in $A_q$. In addition, there is no arc leading from $A_q$ to $D_i$ for $i < p - 1$ and $A_j \rightarrow A_q$ for $j < q$. Hence $A_q$ is the terminal strong component of $D - v$, a contradiction to the choice of $u$.

So $v$ is the only vertex of $D_p$. Since $v$ is a separating vertex of $D$, it follows that $u$ has no in-neighbor in $D_{p-1}$. Since $D$ is not round-decomposable, there is an arc from $D_r$ to $u$ for an index $r \leq p - 2$. Hence $D_i \rightarrow D_j$ for every pair $i, j$ of indices with $r \leq i < j$ and $u \rightarrow D_{p-1}$. By Lemma 8.12.(b) we conclude that $|V(D_{p-1})| = 1$. But then the terminal strong component of $D - v$ is $D_{p-1}$ and the sole vertex of $D_{p-1}$ is not a separating vertex of $D$. □

By symmetry the following propositions hold.

**Lemma 8.16.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $u \not\rightarrow D_1$, then $D_1 - w$ is not strong for every out-neighbor $w$ of $u$ in $D_1$. Let $B_1, B_2, \ldots, B_q$ be the strong decomposition of $D_1 - w$, where $q \geq 2$. Then*

*(a)* $D_i \rightarrow D_j$ *for every pair of indices* $1 \leq i < j \leq p$;

*(b)* $|V(D_i)| = 1$ *for every index* $i$ *with* $2 \leq i \leq p - 1$;

*(c)* $V(B_i) = \{b_i\}$ *for every index* $i > 1$;

*(d)* $u \not\rightarrow b_i$ *for every index* $i$ *with* $2 \leq i \leq q - 1$;

*(e) if* $b_{q-1} \rightarrow w$, *then* $u \not\rightarrow b_q$;

*(f) if* $x \rightarrow b_2$, *then* $|V(B_1)| = 1$;

*(g) if* $x \rightarrow b_i$ *for an index* $i > 1$, *then* $u$ *has no out-neighbors in* $B_1$;

*(h)* $|N^+(u, B_1)| \leq 1$ *and if* $|N^+(u, B_1)| = 1$, *then* $N^+(w, B_1) = N^+(u, B_1)$.

**Lemma 8.17.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $u \not\rightarrow D_1$, then $V(D_1)$ can be partitioned in $W^+ = N^+(u, D_1) \neq \emptyset$, $W^- = N^-(u, D_1) \neq \emptyset$ and $S = V(D_1) - W^+ - W^-$ such that* *(a)* $W^- \to S \to W^+$;

*(b) if* $S = \emptyset$, *then* $|W^+| \leq 2$ *and*

*(i) if* $W^+ = \{w_1, w_2\}$ *such that* $w_1 \to w_2$, *then* $w_2$ *is the only out-neighbor of* $w_1$ *in* $D_1$, *the vertices are consecutive on every Hamiltonian cycle of* $D_1$ *and* $w_2$ *is a separating vertex of* $D_1$ *and* $D$;

*(ii) if* $W^+ = \{w_2\}$, *then* $w_2$ *is a separating vertex of* $D_1$ *and* $D$;

*(c) if* $S \neq \emptyset$, *then* $D[W^-]$ *is transitive,* $|W^+| = 1$ *and* $D[S]$ *is transitive or a* 3*-cycle. Furthermore* $D_p$ *consists of a single vertex.*

**Lemma 8.18.** *Let* $D$ *be a strong local tournament on* $n \geq 6$ *vertices that is not* 2*-connected and not cycle complementary. If* $D$ *is not round-decomposable, then* $D$ *has a separating vertex* $u$ *such that the initial strong component* $D_1$ *of the strong decomposition* $D_1, D_2, \ldots, D_p$ *of* $D - u$, *where* $p \geq 2$, *contains no separating vertex of* $D$.

## 8.2 The number of separating vertices in local tournaments that are not cycle complementary

In this section we use the structural results of Section 8.1 to answer the question how many separating vertices can there possibly be in a local tournament that is not cycle complementary. Firstly we consider round-decomposable local tournaments.

**Remark 8.19.** *Let* $n \geq 5$ *be an odd integer. If* $D$ *is a member of* $\mathcal{R}_n^2$, *it is a* 2*-connected local tournament that is not cycle complementary. Hence every subdigraph of* $D$ *is not cycle complementary. It follows that for every integer* $c$ *with* $1 \leq c \leq n$ *there exists a round local tournament* $D$ *that is not cycle complementary with exactly* $c$ *separating vertices which are, in general, not consecutive on a Hamiltonian cycle of* $D$.

In the next result we address local tournaments that are not round-decomposable.

**Theorem 8.20.** *Let* $D$ *be a strong local tournament that is not cycle complementary and let* $c$ *be the number of separating vertices of* $D$. *If* $D$ *is not round-decomposable, then* $D$ *has a Hamiltonian cycle* $C$ *such that all separating vertices of* $D$ *are consecutive on* $C$. *In addition, if* $c \geq 4$, *then* $C$ *can be partitioned into two consecutive segments* $Q$ *and* $L$ *such that* *(a) $Q$ induces a transitive tournament;*

*(b) $L$ induces a tournament that is isomorphic to $Q_{|L|}$;*

*(c) all separating vertices of $D$ are consecutive on $L$;*

*(d) $|L| = c, c+1$ or $c+2$.*

*Proof.* If $D$ is 2-connected, it has $c = 0$ separating vertices and there is nothing to show. So assume that $D$ is strong, but not 2-connected, and let $u$ be a separating vertex of $D$. Let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$, such that $D_1$ contains no separating vertex of $D$.

By Lemma 8.12.(c) it follows that $|V(D_1)| \leq 3$. If $|V(D_1)| = 1$, we obtain $N^+(u, D_2) \neq \emptyset$ and if $|V(D_1)| = 3$, we obtain $u \to D_1$, since $D_1$ contains no separating vertex of $D$. In the former case let $L_1 = \emptyset$ and in the latter case let $L_1 = \{w\}$, where $w$ is an arbitrary vertex of $D_1$.

We consider two cases depending on the structure of $D$.

*Case 1:* Suppose that $D_p \to u$. Note that a vertex $y$ of $D_p$ is a separating vertex of $D$ if and only if it is the only vertex of $D_p$ and $u$ has no in-neighbor in $D_{p-1}$. Since $D$ is not round-decomposable, there exist two indices $r \leq s$ such that $N^-(u, D_r) \neq \emptyset$ and $N^+(u, D_s) \neq \emptyset$. Let $r$ and $s$ be chosen minimal and maximal, respectively.

If $r = s$, it follows that $2 \leq r \leq p - 1$ and $|V(D_r)| \geq 3$. By Lemma 8.12.(c) we conclude that there is no arc between every pair $D_i, D_j$ of components with $i < r < j$. Hence $D_i \to u \to D_j$ for every index $i < r < j$. Therefore $D_i \to D_j$ for every pair $i, j$ of indices with $i < j \leq r$ or $r \leq i < j$. Now it is easy to see that $u$ is the only separating vertex of $D$.

If $r < s$, it follows that $2 \leq r < s \leq p - 1$. Then $D_i \to D_j$ for every pair $i, j$ of indices with $i < j \leq s$ or $r \leq i < j$. So $D$ has no separating vertices in $D_i$ for $i = 1, 2, \ldots, p - 1$ which implies that $c \leq 2$.

*Case 2:* Suppose that $D_p \not\to u$. Then $D_p$ has the structure as described in Lemmas 8.13 and 8.14.

*Subcase 2.1:* Suppose that $R = \emptyset$. We shall prove the proposition by induction on $|V(D_p)|$.

**Induction basis.** Suppose that $|V(D_p)| = 3$. Let $C = v_1v_2v_3v_1$ be the Hamiltonian cycle of $D_p$. Then either $v_1 \to u \to \{v_2, v_3\}$ or $\{v_1, v_2\} \to u \to v_3$. In the former case $v_1$ and $v_2$ are the separating vertices of $D$ in $D_p$ and in the latter case $v_1$ is the separating vertex of $D$ in $D_p$. So let $L_p = \{v_1, v_2\}$ and $L_p = \{v_1\}$, respectively. Then $L_p$ and $Q_p = V(D_p) - L_p$ from a partition of $D_p$ such that $Q_p$ is a transitive tournament, $L_p$ is a tournament that is isomorphic to $Q_{|L_p|}$, all separating vertices of $D$ in $D_p$ are consecutive on $L_p$, $|L_p| = c'$ or $c' + 1$, where $c'$ is the number of separating vertices of $D$ in $D_p$, the last vertex of $L_p$ dominates $u$ and if $|L_p| = c' + 1$, the first vertex of $L_p$ is not a separating vertex of $D$. Hence $L = L_p \cup \{u\} \cup L_1$ and $Q = V(D) - L$ fulfill the conditions (a)-(d).

**Inductive step.** Suppose that $|V(D_p)| \geq 4$. Then $x_1$ is a separating vertex of $D_p$ and $D$ by Lemma 8.14.(b) and $D_p - x_1$ has the structure given in Lemma 8.13. Let $a_1, a_2, \ldots, a_{q-1}, A_q$ be the strong decomposition of $D_p - x_1$, where $q \geq 2$. Then $|V(A_q)| < |V(D_p)|$. Since $x_1$ is a separating vertex of $D$, it follows that $u \to A_q$. In addition $u \to a_i$ for every index $i$ with $2 \leq i \leq q-1$ by Lemma 8.13.(d). Note that $D - a_i$ is strong for every index $i < q$.

If $|V(A_q)| = 1$, the only vertex $a_q$ of $A_q$ is a separating vertex of $D$ if and only if $x \to a_{q-1}$. If $D - a_q$ is strong, let $L_p = \{x_1\}$ and if $D - a_q$ is not strong, let $L_p = \{a_q, x_1\}$.

If $|V(A_q)| = 3$ and $A_q$ contains no separating vertex of $D$, let $L_p = \{x_1, v\}$, where $v$ is an arbitrary vertex of $A_q$.

Otherwise, by the induction hypothesis, the component $A_q$ can be partitioned in $Q'$ and $L'$ such that $Q'$ is a transitive tournament, $L'$ is a tournament that is isomorphic to $Q_{|L'|}$, all separating vertices of $D$ in $A_q$ are consecutive on $L'$, $|L'| = c'$ or $c' + 1$, where $c'$ is the number of separating vertices of $D$ in $A_q$, the last vertex of $L'$ dominates $x_1$ and if $|L'| = c' + 1$, the first vertex of $L'$ is not a separating vertex of $D$. Let $L_p = L' \cup \{x_1\}$. Then $L = L_p \cup \{u\} \cup L_1$ and $Q = V(D) - L$ fulfill the conditions (a)-(d).

*Subcase 2.2:* Suppose that $R \neq \emptyset$. It follows that $u \to D_i$ for $i < p$. Hence $D$ has no separating vertices in $D_i$ for $i < p$. We observe that the only vertex $x_1$ of $U^-$ is a separating vertex of $D$. Moreover, a vertex $z \in R$ is a separating vertex of $D$ if and only if $R = \{z\}$. Furthermore, it is easy to see that $D - y_i$ is strong for every vertex $y_i \in U^+$. Therefore $c \leq 3$ and the set of separating vertices of $D$ induces a path of order $c$ in $D$ which is part of every Hamiltonian cycle of $D$. □

From the proof of the theorem above we make the following observations.

**Remark 8.21.** *In the situation of Theorem 8.20 the following propositions hold.*

*(a) If $|L| = c + 2$, all vertices of $D[L]$ are separating vertices of $D$ except the first and last vertex and if $|L| = c + 1$, all vertices of $D[L]$ are separating vertices of $D$ except either the first or the last vertex;*

*(b) $D$ has at most $\frac{n+1}{2}$ separating vertices;*

*(c) The in-degree of the first vertex of $D[Q]$ and the out-degree of the last vertex of $D[Q]$ is at most two.*

## 8.3 Partitioning a strong local tournament into two cycles

Using the results of Section 8.1, we characterize all strongly connected local tournaments that are cycle complementary and have the following structure: $D$ has a separating vertex $u$ such that $u$ dominates the initial strong component and is dominated by the terminal strong component of $D - u$. Note that this characterization includes the class of strongly connected, round-decomposable local tournaments that are not 2-connected.

**Theorem 8.22.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not 2-connected. Let $u$ be a separating vertex of $D$ and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $D_p \rightarrow u \rightarrow D_1$, then $D$ is cycle complementary if and only if one of the following conditions is satisfied.*

*(a) $|V(D_i)| \geq 4$ for an arbitrary index $i$;*

*(b) $|V(D_1)| = 3$ and $u$ has an out-neighbor in $D_2$ or $|V(D_p)| = 3$ and $u$ has an in-neighbor in $D_{p-1}$;*

*(c) $|V(D_i)| = 3$ for an index $i$ with $2 \leq i \leq p-1$ and there is an arc from $D_r$ to $D_s$ for a pair $r, s$ of indices with $r < i < s$.*

*Proof.* If $D$ satisfies one of the conditions, it is cycle complementary by Theorem 8.12. So assume that none of the conditions is satisfied. Then $|V(D_i)| \leq 3$ for all indices $i$ and $D - D_i$ is not strong for every component $D_i$ with $|V(D_i)| = 3$. Hence, if $C$ is a cycle in $D$ that contains vertices of more than one component, it contains at least one vertex of every component $D_i$ with $|V(D_i)| = 3$. Therefore $D - C$ is an acyclic local tournament and thus $D$ is not cycle complementary. □

Now we can prove a necessary condition for a local tournament to be not cycle complementary in terms of minimal degree.

**Theorem 8.23.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Then $\delta^+(D) \leq 2$ and $\delta^-(D) \leq 2$ unless $D$ is isomorphic to $T_7^1$.*

*Proof.* If $D$ is a member of $\mathcal{R}_n^2$, it is 2-regular and hence $\delta^+(D) = \delta^-(D) = 2$. So assume that $D$ is not a member of $\mathcal{R}_n^2$. Then $D$ has a separating vertex $u$ by Theorem 8.3. If $D$ is round-decomposable, it follows that $D_p \to u \to D_1$. Hence $d^+(v) \leq 2$ and $d^-(w) \leq 2$ for every vertex $v$ of $D_p$ and $w$ of $D_1$ by Theorem 8.22. If $D$ is not round-decomposable, it has the structure as described in Theorem 8.20. By Remark 8.21 we obtain $d^+(v) \leq 2$ for the last vertex $v$ of $Q$ and $d^-(w) \leq 2$ for the first vertex $v$ of $Q$. □

The following result as well as Corollary 8.6 are immediate by Theorem 8.23.

**Corollary 8.24.** *Let $D$ be a strong local tournament on $n \geq 6$ vertices that is not isomorphic to $T_7^1$. If $\max\{\delta^+(D), \delta^-(D)\} \geq 3$, then $D$ is cycle complementary.*

## 8.4 Partitioning a strong local tournament into $k$ cycles

In this section we use the structural results obtained in Sections 8.1 and 8.2 to give a sufficient criterion for a local tournament to have a $k$-cycle factor. The main result of Section 8.3 serves as our induction basis. Note that Theorem 8.7 is an immediate corollary of Theorem 8.25.

**Theorem 8.25.** *Let $D$ be a strong local tournament on $n$ vertices that is not isomorphic to $T_7^1$ and let $k \geq 2$ be a positive integer. If*

$$\delta^+(D) + \delta^-(D) \geq \frac{k-2}{k-1}n + 3k - 1,$$

*then $D$ has a $k$-cycle factor.*

*Proof.* If $k = 2$, it follows that

$$\delta^+(D) + \delta^-(D) \geq \frac{2-2}{2-1}n + 3 \cdot 2 - 1 = 5$$

which implies that $\max\{\delta^+(D), \delta^-(D)\} \geq 3$. By Theorem 8.23 we conclude that $D$ is cycle complementary, i.e. $D$ has a 2-cycle factor.

Now let $k \geq 3$ be the smallest integer such that $D$ cannot be partitioned into $k$ cycles. Then $D$ can be partitioned into $k - 1 \geq 2$ cycles. Let $C_1, C_2, \ldots, C_{k-1}$ be a partition of $D$ in $k - 1$ cycles such that

**A.** $|V(C_1)| \geq |V(C_2)| \geq \ldots \geq |V(C_{k-1})|$;

**B.** $|V(C_1)|$ is maximal.

Then none of the digraphs $C_i$ is cycle complementary. Since $\delta^+(D) \leq \frac{n-1}{2}$ and $\delta^-(D) \leq \frac{n-1}{2}$, it follows that

$$n - 1 \geq \delta^+(D) + \delta^-(D) \geq \frac{k-2}{k-1}n + 3k - 1$$

which implies that

$$n \geq 3k(k-1).$$

Since $k \geq 3$, it follows particularly that $n \geq 9(k-1)$ and thus, regarding condition A,

$$|V(C_1)| \geq \frac{n}{k-1} \geq 9.$$

We partition $C_2, C_3, \ldots, C_{k-1}$ into two sets by defining

$$\mathcal{F} = \{C_i \mid i \geq 2 \text{ and } |V(C_i)| \geq 6 \text{ and } C_i \ncong T_7^1\} \text{ and}$$
$$\mathcal{H} = \{C_i \mid i \geq 2 \text{ and } |V(C_i)| \leq 5 \text{ or } C_i \cong T_7^1\}.$$

Firstly we prove the following claim.

**Claim 1.** Let $C_i$ be an arbitrary cycle, where $i \geq 2$. If $v$ is not a separating vertex of $C_i$, then either $v \to C_1$ or $C_1 \to v$ or there are no arcs between $v$ and $C_1$.

*Proof of Claim.* Let $C_i$ be an arbitrary cycle, where $i \geq 2$, and let $v$ be a non-separating vertex of $C_i$.

If there are no arcs between $v$ and $C_1$, there is nothing to show.

So suppose that $v$ has a positive neighbor on $C_1$. By Lemma 1.12, either $v \to C_1$ or $C_1$ can be extended by $v$. Since the latter yields a contradiction to the choice of $C_1$, it follows that $v \to C_1$.

By symmetry we conclude that $C_1 \to v$ if $v$ has a negative neighbor on $C_1$ and the proof of this claim is complete. □

Let $u$ and $v$ be two vertices of $C_1$ such that $d^-(u, C_1)$ and $d^+(v, C_1)$ are minimal. By Theorem 8.23 we obtain $d^-(u, C_1) \leq 2$ and $d^+(v, C_1) \leq 2$. We shall show now that

$$d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i)| + 3$$

for every $i = 2, 3, \ldots, k-1$.

*Case 1:* Let $C_i \in \mathcal{F}$ be an arbitrary cycle and let $S_i$ be the set of separating vertices of $C_i$. Note that by Claim 1 every vertex $x \notin S_i$ of $C_i$ contributes at most one to the sum of $d^-(u, C_i)$ and $d^+(v, C_i)$. It follows that

$$d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i) - S_i| + d^-(u, S_i) + d^+(v, S_i).$$

Thus, if $d^-(u, S_i) \leq 3$ or $d^+(v, S_i) \leq 3$ or $|S_i| \leq 3$, the desired inequality is trivially true. So assume that $|S_i| \geq 4$ and $d^-(u, S_i)+d^+(v, S_i) \geq |S_i|+4$. Then $S_i$ induces a sub-tournament of order at least four in $D$. Since every round-decomposable local tournament has the property that every quadruple of separating vertices does not induce a tournament, it follows that $C_i$ is not round-decomposable. Then $D[V(C_i)]$ has the structure described in Theorem 8.20. Let $P = x_1x_2 \ldots x_s$ be the Hamiltonian path of $D[S_i]$ such that $x_q \rightarrow x_r$ for $q > r+1$ and $P$ is a segment of $C_i$. Let $x_0$ be the predecessor of $x_1$ on $C_i$ and let $x_{s+1}$ be the successor of $x_s$ on $C_i$. Since $C_i$ is a local tournament that is not cycle complementary, we deduce that $\{x_4, x_5, \ldots, x_s\} \rightarrow x_0$ and $x_{s+1} \rightarrow \{x_0, x_1, \ldots, x_{s-3}\}$. Using the local tournament property of $D$ and the fact that $d^-(u, S_i) \geq 4$ and $d^+(v, S_i) \geq 4$, we conclude that $v, x_0$ and $u, x_{s+1}$ are adjacent. By Claim 1 we deduce that either $x_0 \rightarrow C_1$ or $C_1 \rightarrow x_0$ and that either $x_{s+1} \rightarrow C_1$ or $C_1 \rightarrow x_{s+1}$. Let $C_i[x_{s+1}, x_0] = y_1y_2 \ldots y_t$.

*Subcase 1.1:* Suppose that $v \rightarrow x_{s+1}$ and $x_0 \rightarrow u$. Then $x_0 \rightarrow C_1 \rightarrow x_5$. It follows that

$$C = vC_i[x_{s+1}, x_0]C_1[v^+, v] \text{ and } x_1x_2 \ldots x_sx_1$$

are complementary cycles in $D[V(C_1) \cup V(C_i)]$ such that $|V(C)| > |V(C_1)|$, a contradiction.

*Subcase 1.2:* Suppose that $x_{s+1} \rightarrow v$ and $x_0 \rightarrow u$. Then $\{x_0, x_{s+1}\} \rightarrow C_1$ and

$$C = C_1[v^+, v]C_i[x_s, x_0]v^+ \text{ and } x_1x_2 \ldots x_{s-1}x_1$$

are complementary cycles in $D[V(C_1) \cup V(C_i)]$ such that $|V(C)| > |V(C_1)|$, again a contradiction.

By symmetry the case that $v \rightarrow x_{s+1}$ and $u \rightarrow x_0$ can be solved analogously.

*Subcase 1.3:* Suppose that $x_{s+1} \rightarrow v$ and $u \rightarrow x_0$. Then $x_5 \rightarrow C_1 \rightarrow x_0$. We consider two subcases.

*Subcase 1.3.1:* Suppose that there exists a pair $j_1, j_2$ of indices with $1 \leq j_1 \leq j_2 - 4 \leq s - 4$ such that $x_{j_1} \rightarrow u$ and $v \rightarrow x_{j_2}$. Then the cycles

$$C = x_{j_1}C_1[u, v]C_i[x_{j_2}, x_{j_1}] \text{ and } C_i[x_{j_1+1}, x_{j_2-1}]x_{j_1+1}$$

are vertex disjoint in $D[V(C_1) \cup V(C_i)]$.

If $v^+ = u$, these cycles are complementary in $D[V(C_1) \cup V(C_i)]$ and $|V(C)| > |V(C_1)|$, a contradiction. So assume that $v^+ \neq u$ and let $C_1[v^+, u^-] = w_1 w_2 \dots w_r$, where $r \geq 1$.

If $C$ can be extended by $w_1, w_2, \dots, w_r$ to a cycle $C'$, the cycles

$$C' \text{ and } C_i[x_{j_1+1}, x_{j_2-1}]x_{j_1+1}$$

are complementary cycles of $D[V(C_1) \cup V(C_i)]$ such that $|V(C')| > |V(C_1)|$, again a contradiction.

Hence, there is an index $q$ with $1 \leq q \leq r$ such that $w_q \to C$ in view of Lemma 1.12. By Claim 1 it follows that $C_1 \to (C_i - S_i)$. But then

$$C^* = x_{j_1} C_1[u, u^-] C_i[x_{s+1}, x_{j_1}] \text{ and } C[x_{j_1+1}, x_s]x_{j_1+1}$$

are complementary in $D[V(C_1) \cup V(C_i)]$ such that $|V(C^*)| > |V(C_1)|$, the final contradiction.

*Subcase 1.3.2:* Suppose that there exists a pair $j_1, j_2$ of indices with $j_1 + j_2 = s + 4$ such that $N^-(u, S_i) = \{x_{s-j_1+1}, x_{s-j_1+2}, \dots, x_s\}$ and $N^+(v, S_i) = \{x_1, x_2, \dots, x_{j_2}\}$. Since $u \to y_t$, it follows by Lemma 1.12 that there is an index $r$ with $1 \leq r \leq p-1$ such that $y_r \to u \to y_{r+1}$. Analogously, since $y_1 \to v$, there is an index $q$ with $1 \leq q \leq p-1$ such that $y_q \to v \to y_{q+1}$. Note that $q \leq r$ by Claim 1.

If $q < r$, the cycles

$$C = vC_i[y_{q+1}, y_r]C_1[v^+, v] \text{ and } C_i[y_{r+1}, y_q]y_{r+1}$$

are complementary in $D[V(C_1) \cup V(C_i)]$ such that $|V(C)| > |V(C_1)|$, a contradiction.

Otherwise $q = r$. Let $x = x_{s-j_1+2}$. If $x \to y_j$ for an index $j$ with $j \leq r$, let

$$C' = vC_i[x^+, y_{j-1}]C_1[v^+, v] \text{ and } xC_i[y_j, x].$$

These cycles are complementary in $D[V(C_1) \cup V(C_i)]$ and $|V(C')| > |V(C_1)|$, again a contradiction. So assume that $y_j \to x$ for all indices $j$ with $j \leq r$. It follows that

$$C^* = x^- C_1[u, u^-]C_i[y_{r+1}, x^-] \text{ and } C_i[x, y_r]x$$

are complementary cycles of $D[V(C_1) \cup V(C_i)]$ such that $|V(C^*)| > |V(C_1)|$, the final contradiction.

*Case 2.* Let $C_i \in \mathcal{H}$ be an arbitrary cycle. Let $R_i = \{w \in V(C_i) \mid v \to w \to u\}$. We will show that $|R_i| \leq 3$.

If $|V(C_i)| = 3$, the proposition is immediate.

If $|V(C_i)| = 4$ or $|V(C_i)| = 5$, assume that $|R_i| \geq 4$. Then $d^-(u, C_i) + d^+(v, C_i) \geq |V(C_i)| + 4$. It follows that $v \to C_i$ or $C_i \to u$ and $C_i$ induces a strong tournament $T_i$ in $D$. By Corollary 5.1 there exist two non-separating vertices in $T_i$. But then at least one of these vertices has an in- as well as an out-neighbor on $C_1$, a contradiction to Claim 1. Hence $|R_i| \leq 3$.

If $C_i$ is isomorphic to $T_7^1$, the tournament $D[V(C_i)]$ is 2-connected and thus, $R_i = \emptyset$.

By the cases above, it follows that $d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i)| + 3$ for every index $i = 2, 3, \ldots, k-1$. Therefore

$$\begin{aligned}
\delta^+(D) + \delta^-(D) &\leq 4 + \sum_{i=2}^{k-1}(|V(C_i)| + 3) \\
&= 4 + \sum_{i=2}^{k-1}|V(C_i)| + \sum_{i=2}^{k-1} 3 \\
&\leq 4 + (n - n_1) + 3(k-2) \\
&\leq \frac{k-2}{k-1}n + 3k - 2,
\end{aligned}$$

a contradiction to our assumption. This contradiction completes the proof of this theorem. □

# Chapter 9

# Extendable cycles in in-tournaments

In 1989, Hendry [44] defined a digraph $D$ to be *cycle extendable* if $D$ contains a cycle and every non-Hamiltonian cycle of $D$ is extendable. Furthermore, $D$ is *fully cycle extendable* if $D$ is cycle extendable and every vertex of $D$ belongs to a cycle of length three. Tewes and Volkmann [69] extended these definitions in considering only cycles whose length exceed a certain bound $k$.

**Definition 9.1** (Tewes & Volkmann [69] 2001)**.** *Let $D$ be a digraph of order $n$ and let $3 \leq k \leq n$ be an integer. Then $D$ is* $k$-extendable *if every cycle of length $t$, where $k \leq t < n$, is extendable. Moreover $D$ is called* fully $k$-extendable *if $D$ is $k$-extendable and every vertex of $D$ belongs to a cycle of length $k$.*

**Remark 9.2.** *The terms* fully 3-extendable *and* fully cycle extendable *describe the same property. Obviously, every fully $k$-extendable digraph is vertex $k$-pancyclic.*

Concerning cycle-extendibility Volkmann and Tewes [69] proved the following result.

**Theorem 9.3** (Tewes & Volkmann [68] 1998)**.** *Let $D$ be a connected in-tournament of order $n$ such that $\delta(D) \geq 1$. Then $D$ is $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

Using Theorem 9.3, Volkmann and Tewes [69] showed that the following two degree conditions are sufficient for a strong in-tournament to be fully $k$-extendable for large values of $k$.

**Theorem 9.4** (Tewes & Volkmann [69] 2001)**.** *Let $D$ be a strong in-tournament of order $n \geq 6$ with $\delta(D) \geq 2$. Then every vertex of $D$ is contained in a cycle of length $n - 3$.*

**Theorem 9.5** (Tewes & Volkmann [69] 2001)**.** *Let $D$ be a strong in-tournament of order $n$ with* $\max\{\delta^-(D), \delta^+(D)\} \geq 2$*. Then every vertex of $D$ is contained in a cycle of length $n-2$.*

Then the following results are immediate corollaries.

**Corollary 9.6** (Tewes & Volkmann [69] 2001)**.** *Every strong in-tournament $D$ of order $n$ with $\delta(D) \geq 2$ is fully $(n-3)$-extendable.*

**Corollary 9.7** (Tewes & Volkmann [69] 2001)**.** *Every strong in-tournament $D$ of order $n$ with* $\max\{\delta^-(D), \delta^+(D)\} \geq 2$ *is fully $(n-2)$-extendable.*

Tewes and Volkmann proved that the following extension of Theorem 9.3 is valid for special values of $\delta$.

**Theorem 9.8** (Tewes & Volkmann [69] 2001)**.** *Every strong in-tournament $D$ of order $n$ with $\delta(D) = 2$ or $\delta(D) > \frac{8n-17}{31}$ is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

For $\delta(D) = 1$ we have $n - \lfloor \frac{4\delta+1}{3} \rfloor = n-1$ and there are several examples, e.g. the cycle on $n$ vertices, which illustrate that not every strongly connected in-tournament with minimum degree 1 is fully $(n-1)$-extendable. But Tewes and Volkmann stated the following conjecture.

**Conjecture 9.9** (Tewes & Volkmann [69] 2001)**.** *Let $D$ be a strongly connected in-tournament of order $n$ such that $3 \leq \delta(D) \leq \frac{8n-17}{31}$. Then $D$ is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

In order to proof Conjecture 9.9, we shall show that every vertex of a strongly connected in-tournament $D$ with $3 \leq \delta(D) \leq \frac{8n-17}{31}$ belongs to a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$.

The following result plays an important role in our investigation.

**Lemma 9.10** (Tewes & Volkmann [69] 2001)**.** *Let $D$ be an in-tournament and let $C = u_1u_2 \dots u_su_1$, where $s \geq 4$, be a cycle in $D$. If $u_i \to u_s$ for an index $2 \leq i \leq s-2$, then either $D[V(C) - u_m]$ is strong for an index $2 \leq m \leq s-1$ or $u_{s-1} \to u_j$ for every index $j$ with $1 \leq j \leq i$.*

## 9.1 Extending cycles and deleting vertices

In this section we investigate the cycle structure of strong in-tournaments. In particular we analyze how cycles can be transformed into other cycles by adding, deleting or exchanging vertices. The first lemma shows how we can transform a given cycle into a longer cycle.

**Lemma 9.11.** *Let $D$ be a strong in-tournament and let $C$ be a cycle of length $n-k-1$ in $D$, where $0 \leq k \leq \frac{n}{2}-1$. If $|N^+(x,C)| \geq 1$ for a vertex $x \notin V(C)$, then $x$ belongs to a cycle of length $n-k$.*

*Proof.* Let $C = u_1u_2\dots u_su_1$ be a cycle of length $s = n-k-1$ in $D$ and let $x \notin V(C)$ be a vertex that has at least one positive neighbor on $C$. Then either $x \to C$ or there exists an index $i$ such that $1 \leq i \leq s$ and $u_i \to x \to u_{i+1}$ by Theorem 1.25.

If $u_i \to x \to u_{i+1}$ for an index $1 \leq i \leq s$, the cycle $C$ can be extended by $x$ and thus, $x$ belongs to a cycle of length $n-k$ in $D$.

So assume that $x \to C$. Since $D$ is strong, there exists a path leading from $C$ to $x$. Let $P$ be a shortest path leading from $C$ to $x$ and let, without loss of generality, $V(C) \cap V(P) = \{u_s\}$. Note that $r = |V(P)| \leq n-s+1 = k+2 \leq n/2+1$. Then

$$P[u_s,x]C[u_{r-1},u_s]$$

is a cycle of length $r-1+s-(r-2) = s+1 = n-k$ that includes $x$ which completes the proof of this lemma. □

The next result shows how we can transform a given cycle into a cycle of the same length.

**Lemma 9.12.** *Let $D$ be a strong in-tournament and let $C$ be a cycle of length $n-k$ in $D$, where $0 \leq k \leq \frac{n}{2}$. If $|N^+(x,C)| \geq 2$ and $d^-(x) \geq 2$ for a vertex $x \notin V(C)$, then $x$ belongs to a cycle of length $n-k$.*

*Proof.* Let $C = u_1u_2\dots u_su_1$ be a cycle of length $s = n-k$ in $D$.

If $x \to C$, there exists a path leading from $C$ to $x$, since $D$ is strong. Let $P$ be a shortest such path and let, without loss of generality, $V(C) \cap V(P) = \{u_s\}$. Since $d^-(x) \geq 2$, it follows that $r = |V(P)| \leq n-s = k \leq n/2$. Then

$$P[u_s,x]C[u_r,u_s]$$

is a cycle of length $r-1+s-(r-1) = s = n-k$ that includes $x$.

So assume that $x \not\to C$. In view of Theorem 1.25 there exists an index $i$ such that $u_i \to x \to u_{i+1}$, say $i = s$. Since $|N^+(x,C)| \geq 2$, the vertex $x$ has a positive neighbor on $C$ besides $u_s$. If $x \to u_2$,

$$xC[u_2,u_s]x$$

is a cycle of length $n-k$ through $x$. Hence assume that $x \not\to u_2$. Using Theorem 1.25, it follows that there exists an index $2 \leq j \leq s-2$ such that

$u_j \to x \to u_{j+1}$. Then $u_s$ and $u_j$ are adjacent, say $u_j \to u_s$. In view of Lemma 9.10, it follows that $D[V(C)-u_m]$ is strong for an index $2 \leq m \leq s$ and thus, according to Theorem 1.26, Hamiltonian. Let $C'$ be a Hamiltonian cycle of $D[V(C)-u_m]$ and note that $|N^+(x, C')| \geq 1$ and $|N^-(x, C')| \geq 1$. By applying Theorem 1.25 to $x$ and $C'$, we find a cycle of length $n-k$ through $x$ which completes the proof of this lemma. □

The final result in this section shows how we can transform a given cycle into a shorter cycle.

**Lemma 9.13.** *Let $D$ be a strong in-tournament with minimal degree $\delta = \delta(D)$ such that $3 \leq \delta \leq \frac{8n-17}{31}$ and let $C$ be a cycle of length $n-k+1$ in $D$, where $1 \leq k \leq \lfloor \frac{4\delta+1}{3} \rfloor$. If $|N^+(x, C)| \geq 1$ for a vertex $x \notin V(C)$, then $x$ belongs to a cycle of length $n-k$.*

*Proof.* Let $C = u_1 u_2 \ldots u_s u_1$ be a cycle of length $s = n-k+1$ in $D$.

Firstly we shall show that $C$ has a chord, i.e. $u_i \to u_j$ for some $|i-j| \geq 2$. Assume to the contrary that $C$ does not have a chord. It follows that every external vertex $z$ has at most two in-neighbors on $C$. In addition, since $\delta \geq 3$, we conclude that every vertex of $C$ dominates at least two external vertices. Hence

$$2(n-k+1) = 2s \leq |\{xy \in E \mid x \in V(C), y \notin V(C)\}| \leq 2(n-s) = 2(k-1)$$

which leads to

$$n \leq 2k-2 \leq 2\lfloor \frac{4\delta+1}{3} \rfloor - 2 \leq 2\lfloor \frac{4\frac{8n-17}{31}+1}{3} \rfloor - 2 < n,$$

a contradiction.

So assume that $C$ has a chord. Let, without loss of generality, $u_i \to u_s$ for an index $i$ with $2 \leq i \leq s-2$. Applying Lemma 9.10, we conclude that $D[V(C)-u_m]$ is strong for an index $m$ with $2 \leq m \leq s$. It follows that there exists a cycle $C'$ of length $n-k$ such that $V(C') \subseteq V(C)$. Let $C' = v_1 v_2 \ldots v_r v_1$ be a cycle of length $r = s-1 = n-k$ such that the distance $d(x, C')$ from $x$ to $C'$ is minimal. Then $d(x, C') \leq 2$. If $|N^+(x, C')| \geq 2$, the vertex $x$ belongs to a cycle of length $n-k$ in view of Lemma 9.12.

Thus assume for the remaining part of the proof that $|N^+(x, C')| \leq 1$. We consider two cases depending on the number of out-neighbors of $x$ on $C'$.

*Case 1.* Suppose that $|N^+(x, C')| = 1$. We may assume, without loss of generality, that $v_r \to x \to v_1$ and $(C - v_1) \rightsquigarrow x$.

If $v_i \to v_r$ for an index $2 \le i \le r-2$, the digraph $D[V(C') - v_t]$ is strong for an index $t$ with $2 \le t \le r$ in view of Lemma 9.10. Since $x \to v_1 \ne v_t$, it follows that $D$ has a cycle $C^*$ of length $n-k-1$ such that $N^+(x, C^*) \ne \emptyset$. By Lemma 9.11 we conclude that $x$ belongs to a cycle of length $n-k$.

So assume that $v_{r-1}$ is the only negative neighbor of $v_r$ on $C'$. Then

$$\begin{aligned} N^+(x) - \{v_1\} &\subseteq V(D) - V(C') - \{x\} \text{ and} \\ N^-(v_r) - \{v_{r-1}\} &\subseteq V(D) - V(C') - \{x\}. \end{aligned}$$

Furthermore, note that

$$\begin{aligned} |V(D) - V(C') - \{x\}| = k - 1 &\le \lfloor \frac{4\delta+1}{3} \rfloor - 1, \\ |N^+(x) - \{v_1\}| &\ge \delta - 1 \text{ and} \\ |N^-(v_r) - \{v_{r-1}\}| &\ge \delta - 1. \end{aligned}$$

Since $\delta \ge 3$ implies that

$$\begin{aligned} |N^+(x) - \{v_1\}| + |N^-(v_r) - \{v_{r-1}\}| &\ge 2\delta - 2 \\ &> \lfloor \frac{4\delta+1}{3} \rfloor - 1 \\ &\ge |V(D) - V(C') - \{x\}|, \end{aligned}$$

there exists a vertex $z \notin V(C')$ such that $x \to z \to v_r$. In addition, it follows that

$$\begin{aligned} |N^-(x, C')| &\ge \delta - [(k-1) - (\delta - 1)] \\ &= 2\delta - k \\ &\ge 2\delta - \lfloor \frac{4\delta+1}{3} \rfloor \\ &> 1. \end{aligned}$$

We consider three subcases.

*Subcase 1.1.* Suppose that $v_{r-1} \to x$. Then

$$xC'[v_1, v_{r-1}]x$$

is a cycle of length $n-k$ through $x$.

*Subcase 1.2.* Suppose that $v_{r-2} \to x$. Then we consider the cycle

$$C^* = xC'[v_1, v_{r-2}]x$$

of length $r-1$. If $v_r \to C^*$, the digraph $D$ contains the cycle

$$xzv_rC'[v_2, v_{r-2}]x$$

of length $n-k$ through $x$. So assume that $v_r$ has at least one positive and one negative neighbor on $C^*$. But then $C^*$ is extendable by $v_r$ which also leads to a cycle of length $n-k$ through $x$.

*Subcase 1.3.* Suppose that $v_i \to x$ for an index $i$ with $2 \le i \le r-3$. In this case we consider the cycle

$$C^* = xzC'[v_r, v_i]x$$

of length $i+3 \le r$ through $x$. Note that none of the vertices $v_j$, where $i < j < r$, dominates $C^*$, since, according to our assumption, $v_{r-1}$ is the only negative neighbor of $v_r$ on $C'$ and $v_{r-1} \not\to x$. Using Theorem 1.25, it follows that we can insert the vertices $v_{r-1}, v_{r-2}, \dots, v_{i+3}$ in $C^*$ resulting in a cycle of length $n-k$ through $x$.

*Case 2.* Suppose that $|N^+(x, C')| = 0$. By choice of $C'$ we obtain that $d(x, C') = 2$. Let $P = xyz$ be a path leading from $x$ to $C'$. Then $|N^+(y, C')| \ge 1$ and following Case 1 we conclude that there exists a cycle of length $n-k$ in $D$ that includes $y$, a contradiction to the choice of $C'$ which completes the proof of this lemma. □

## 9.2 Solution of a conjecture of Tewes and Volkmann regarding cycle extendibility

Using the results of Section 9.1, we show in this section that every vertex of a strongly connected in-tournament of order $n$ with minimal degree $\delta$ such that $3 \le \delta \le \frac{8n-17}{31}$ belongs to a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$. This result in combination with Theorem 9.3 solves Conjecture 9.9 in positive.

**Theorem 9.14.** *Let $D$ be a strong in-tournament of order $n$ with minimal degree $\delta = \delta(D)$ such that $3 \le \delta \le \frac{8n-17}{31}$. Then every vertex of $D$ belongs to a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$.*

*Proof.* Firstly we shall show that there exists a cycle of length $n - \lfloor \frac{4\delta+1}{3} \rfloor$ in $D$. Note that, in view of Theorem 1.26, the in-tournament $D$ has a Hamiltonian cycle, since it is strong. Let $C$ be a shortest cycle in $D$ such that $|V(C)| \ge n - \lfloor \frac{4\delta+1}{3} \rfloor$. Since $\delta \ge 3$, the cycle $C$ has length at most $n-1$. Let $|V(C)| = n-k+1$, where $k \ge 2$, be the length of $C$. If $k \le \lfloor \frac{4\delta+1}{3} \rfloor$, there exists a vertex $x \notin V(C)$ such that $|N^+(x, C)| \ge 1$. In view of Lemma 9.13 this vertex belongs to a cycle of length

$n-k$ in $D$, a contradiction to the choice of $C$. Hence $k = \lfloor\frac{4\delta+1}{3}\rfloor + 1$, i.e. $D$ contains a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$.

We shall show next that every vertex of $D$ belongs to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ in $D$. Assume that $D$ has a vertex $x$ that does not belong to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$. Let $C = u_1u_2\dots u_su_1$ be a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ such that $x$ has minimal distance to $C$ and let $P = x_0x_1\dots x_r$ be a shortest path leading from $x = x_0$ to $C$. Then $x_{r-1} \notin V(C)$ and $N^+(x_{r-1}, C) \neq \emptyset$.

If $|N^+(x_{r-1}, C)| \geq 2$, we conclude by Lemma 9.12 that $x_{r-1}$ belongs to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$, a contradiction to the choice of $C$. So assume that $|N^+(x_{r-1}, C)| = 1$. We may assume, without loss of generality, that $u_s \to x_{r-1} \to u_1$. We consider two cases depending on the number of in-neighbors of $u_s$ on $C$.

*Case 1.* Suppose that $|N^-(u_s, C)| \geq 2$. Then, according to Lemma 9.10, there exists a vertex $u_m \in V(C)$, where $2 \leq m \leq s$, such that $D[V(C) - u_m]$ is strong. Let $C'$ be a Hamiltonian cycle of $D[V(C) - u_m]$. Note that $C'$ has length $n - \lfloor\frac{4\delta+1}{3}\rfloor - 1 \geq \frac{n}{2}$. Furthermore, note that $|N^+(x_{r-1}, C')| \geq 1$. Using Lemma 9.11, we conclude that $x_{r-1}$ belongs to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$, a contradiction to the choice of $C$.

*Case 2.* Suppose that $|N^-(u_s, C)| = 1$. Then $|N^-(u_s, D - C)| \geq \delta - 1$. It follows that

$$|N^+(u_s, C)| \geq \delta - [(\lfloor\frac{4\delta+1}{3}\rfloor - 1) - (\delta - 1)] = 2\delta - \lfloor\frac{4\delta+1}{3}\rfloor > 1$$

and thus, $u_s$ has at least two positive neighbors on $C$. It follows that $C$ has a chord which implies, in view of Lemma 9.10, that $D[V(C) - u_m]$ is strong for an index $1 \leq m \leq s$.

If $D[V(C) - u_m]$ is strong for an index $m \neq 1$, let $C'$ be a Hamiltonian cycle of $D[V(C) - u_m]$. Note that $|N^+(x_{r-1}, C')| \geq 1$. In view of Lemma 9.11 it follows that $x_{r-1}$ belongs to a cycle of length $|V(C')| + 1 = |V(C)|$, a contradiction to the choice of $C$.

So assume that $m = 1$ and that $D[V(C) - u_i]$ is not strong for every index $i > 1$. Then the only in-neighbor of $u_s$ on $C$ is $u_{s-1}$ and the only out-neighbor of $x_{r-1}$ on $C$ is $u_1$. Since $\delta \geq 3$ and $|V(D) - V(C) - \{x_{r-1}\}| = \lfloor\frac{4\delta+1}{3}\rfloor - 1$, it follows that

$$\begin{aligned}|N^+(x_{r-1}) - \{x_{r-1}\}| + |N^-(v_s) - \{x_{r-1}\}| &\geq 2\delta - 2\\ &> \lfloor\frac{4\delta+1}{3}\rfloor - 1\\ &= |V(D) - V(C) - \{x_{r-1}\}|\end{aligned}$$

and thus, there exists a vertex $z \notin V(C)$ such that $x_{r-1} \to z \to v_s$. Moreover, we observe that

$$|N^-(x_{r-1},C')| \geq \delta - (\lfloor\frac{4\delta+1}{3}\rfloor - \delta) = 2\delta - \lfloor\frac{4\delta+1}{3}\rfloor > 1.$$

Hence $x_{r-1}$ has a negative neighbor on $C'$ besides $v_s$. Let $C' = v_2v_3\ldots v_sv_2$ be a Hamiltonian cycle of $D[V(C)-u_1]$ such that $v_s = u_s$. We consider three subcases depending on the negative neighborhood of $x_{r-1}$ on $C'$.

*Subcase 2.1.* Suppose that $v_{s-2} \rightarrow x_{r-1}$. Then

$$x_{r-1}zC'[v_s, v_{s-2}]x_{r-1}$$

is a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ through $x_{r-1}$, a contradiction.

*Subcase 2.2.* Suppose that $v_{s-2} \not\rightarrow x_{r-1}$ and that there exists a vertex $v \neq v_{s-1}$ in $C'$ that dominates $x_{r-1}$. Let $2 \leq t \leq s-3$ be the greatest index such that $v_t \rightarrow x_{r-1}$. We consider the cycle

$$C^* = x_{r-1}zC'[v_s, v_t]x_{r-1}.$$

If $v_{s-1} \not\rightarrow C^*$, the cycle $C^*$ can be extended by $v_{s-1}$. Since $v_{s-2} \not\rightarrow x_{r-1}$ and because of the choice of $t$ none of the vertices $v_{s-2}, v_{s-3}, \ldots, v_{t+2}$ dominates $x_{r-1}$. It follows that we can insert these vertices one after the other in $C^*$, leading to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ that includes $x_{r-1}$.

So assume that $v_{s-1} \rightarrow C^*$. Then $v_2v_3\ldots v_{s-1}v_2$ is a cycle in $D$ and $u_1$ has a positive neighbor on this cycle. If $u_1$ can be inserted in $v_2v_3\ldots v_{s-1}v_2$, the digraph $D[V(C)-u_s]$ is strong, a contradiction. Hence $u_1 \rightarrow \{v_2, v_3, \ldots, v_{s-1}\}$. But then

$$x_{r-1}u_1C'[v_2, v_{s-1}]x_{r-1}$$

is a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ through $x_{r-1}$, again a contradiction.

*Subcase 2.3.* Suppose that $N^-(x_{r-1}, C') = \{v_{s-1}, v_s\}$. Note that $u_1$ has a positive neighbor on $C'$, say $u_1 \rightarrow v_q$, where $2 \leq q \leq s-1$. We consider the cycle

$$C^* = x_{r-1}u_1C'[v_q, v_{s-1}]x_{r-1}.$$

If $q = 2$, the cycle $C^*$ is a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ through $x_{r-1}$. So assume that $q \geq 3$. Due to our assumption none of the vertices $v_{q-1}, v_{q-2}, \ldots, v_2$ dominates $C^*$. Therefore we can insert these vertices one after the other in $C^*$, leading to a cycle of length $n - \lfloor\frac{4\delta+1}{3}\rfloor$ that includes $x_{r-1}$. This final contradiction completes the proof of Subcase 2.3 and of this theorem. □

Combining Theorems 9.3 and 9.14 we are now able to solve Conjecture 9.9 in positive.

**Corollary 9.15.** *Let $D$ be a strong in-tournament of order $n$ such that $3 \leq \delta \leq \frac{8n-17}{31}$. Then $D$ is fully $(n - \lfloor \frac{4\delta+1}{3} \rfloor)$-extendable.*

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# List of notations

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