Upper bounds for the domination number in graphs of diameter two

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Abstract

A vertex set D of a graph G is a dominating set if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. If G is a graph of diameter two and order $n \geq 24$, then we prove in this paper that $\gamma(G) \leq \lfloor n/4 \rfloor$. As an application of this bound, we present partial solutions of problems posed by Dunbar, Haynes and Hedetniemi [3] and Volkmann [15].

Keywords: Domination number; Graphs of diameter two; Minimum degree

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1 Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The *complement* of a graph G is denoted by $\overline{G}$, and the *complement of a set of vertices* $W \subseteq V(G)$ is denoted by $\overline{W} = V(G) - W$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N_G(v)$ of the vertex $v \in V(G)$ consists of the vertices adjacent to v in G, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is defined by

$d_G(v) = |N_G(v)|$. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$ we denote the *minimum degree* and *maximum degree* of the graph G, respectively. We write K_n for the *complete graph* with n vertices and C_p for the *cycle* of length p. The *diameter* of a graph G is denoted by $\mathrm{diam}(G)$. A graph is *P_4-free* if and only if it contains no induced subgraph isomorphic to the path P_4 of order four.

A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the cardinality of any smallest dominating set.

Some upper bounds for the domination number $\gamma = \gamma(G)$ of a graph G, depending on the order $n = n(G)$ and the minimum degree $\delta = \delta(G)$ are $\gamma \leq n/2$ (if $\delta \geq 1$) given by Ore [10] in 1962, and $\gamma \leq 2n/5$ (if $\delta \geq 2$, with the exception of seven graphs) given by McCuaig and Shepherd [8] in 1989. Sharper results for higher minimum degree can be found in [6, 9, 11, 13, 16].

Bounds for the domination number of a graph involving the diameter were given, for example, by Brigham, Chinn and Dutton [1], Goddard and Henning [4], and Dorfling, Goddard and Henning [2]. The first group of authors showed in 1988 that if G is a graph with $\gamma(G) \geq 3$, then the diameter of the complement of G is at most two. Goddard and Henning investigated planar graphs and proved that $\gamma(G) \leq 2$ if G is planar with diameter two with the exception of a single graph on nine vertices. Furthermore, Dorfling, Goddard and Henning showed that $\gamma(G) \leq 9$ if G is planar with diameter three and that every sufficiently large planar graph of diameter three has domination number at most six.

For detailed information on domination and related topics see the comprehensive monograph [5] by Haynes, Hedetniemi and Slater.

In a graph G of diameter two, the open neighborhood of any vertex dominates G, and thus the next upper bound is immediate.

Observation 1 (Haynes, Hedetniemi & Slater [5], p. 55)**.** *If a graph G has diameter two, then $\gamma(G) \leq \delta(G)$.*

Let G be a P_4-free graph. Then the complement of G is also P_4-free and if G has order at least four and is connected, its complement is not connected (see [12]). Hence, there is an even better bound for the class of P_4-free graphs.

Observation 2. *If G is a connected P_4-free graph, then $\gamma(G) \leq 2$.*

Using Observation 1, Hellwig and Volkmann [6] presented an upper bound for the domination number in graphs of diameter two in terms of order.

Theorem 3 (Hellwig & Volkmann [6] 2006)**.** *If G is a graph of order n and diameter two, then $\gamma(G) \leq \lfloor n/4 \rfloor + 1$.*

In addition, they showed the following result which will be useful in the proof of Theorem 5.

Theorem 4 (Hellwig & Volkmann [6] 2006)**.** *Let* G *be a graph of diameter two. If there exists at least one vertex* u *of minimum degree such that* $|N(u) \cap N(v)| \geq 2$ *for all* $v \notin N[u]$*, and* $N(u)$ *is not independent, then* $\gamma(G) \leq \delta(G) - 1$*.*

2 Main results

If the order of the graph of diameter two is large enough, then the following result is stronger than Theorem 3.

Theorem 5. *Let* G *be a graph of order* n *and diameter two. If* $n = 4p + r$ *with integers* $p \geq 1$ *and* $0 \leq r \leq 3$*, then* $\gamma(G) \leq \lfloor n/4 \rfloor = p$ *when* $r = 0$ *and* $p \geq 4$ *or* $r = 1$ *and* $p \geq 5$ *or* $r \in \{2, 3\}$ *and* $p \geq 6$*.*

Proof. If $\delta = \delta(G) \leq \lfloor n/4 \rfloor = p$, then we obtain the desired result by Observation 1.

Now let $\delta = \lfloor n/4 \rfloor + 1 + k = p + 1 + k$ for an integer $k \geq 0$, and let u be an arbitrary vertex of minimum degree in $V(G)$. Furthermore, define $R = V(G) - N[u]$. If $|N(x) \cap N(u)| \geq k + 3$ for each vertex $x \in R$, then $N(u) - \{a_1, a_2, \ldots, a_{k+2}\}$ dominates R for each $k + 2$ different vertices $a_1, a_2, \ldots, a_{k+2} \in N(u)$, and thus $\{u\} \cup (N(u) - \{a_1, a_2, \ldots, a_{k+2}\})$ is a dominating set of G of size p, and we are done.

Now we investigate the case that there exists a vertex $v \in R$ such that $|N(v) \cap N(u)| \leq k + 2$ and thus

$$|N(v) \cap R| \geq \delta - (k + 2) = p - 1.$$

Consequently, v dominates p vertices in R, and thus there exist at most $|R| - p$ vertices, which are not dominated by $\{u, v\}$. Since $\mathrm{diam}(G) = 2$, these vertices can be dominated by at most $\left\lceil \frac{|R|-p}{2} \right\rceil$ vertices. Thus

$$\begin{aligned}
\gamma(G) &\leq 2 + \left\lceil \frac{|R| - p}{2} \right\rceil = 2 + \left\lceil \frac{n - \delta - 1 - p}{2} \right\rceil \\
&= 2 + \left\lceil \frac{n - 2p - 2 - k}{2} \right\rceil = 2 + \left\lceil \frac{2p + r - 2 - k}{2} \right\rceil,
\end{aligned}$$

and so $\gamma(G) \leq p$ when $r = 0$ and $k \geq 2$, or $r = 1$ and $k \geq 3$, or $r = 2$ and $k \geq 4$, or $r = 3$ and $k \geq 5$.

If $|N[v] \cap R| \geq p + r - k + 2$, then $|R - N[v]| \leq 4p + r - (p + 2 + k) - (p + r - k + 2) = 2p - 4$. Since $\mathrm{diam}(G) = 2$, these vertices can be dominated by at most $p - 2$ vertices and thus $\gamma(G) \leq p$.

Now suppose that $|N[v] \cap R| = p + t$ with $0 \le t \le r - k + 1$. Note that $r-k+1 \le 4$. Let $R_1 = R - N[v]$. Then $|R_1| = 4p+r-(p+2+k)-(p+t) = 2p+r-k-t-2$. If there exists a vertex w that dominates at least $r-k-t+4$ vertices in R_1, then $|R_1 - N[w]| \le 2p+r-k-t-2-(r-k-t+4) = 2p-6$. Since $\mathrm{diam}(G) = 2$, these vertices can be dominated by at most $p-3$ vertices and thus $\gamma(G) \le p$.

Hence, each vertex outside of R_1 has at most $r-k-t+3$ neighbors in R_1 (u and v have none) and each vertex in R_1 has at most $r-k-t+2$ neighbors in R_1. It follows that

$$\begin{aligned}(|\overline{R}_1| - 2)(r-k-t+3) &\ge m(\overline{R}_1, R_1) \\ &= m(R_1, \overline{R}_1) \ge |R_1|(\delta - (r-k-t+2))\end{aligned}$$

$$\begin{aligned}\iff\ & (2p+k+t)(r-k-t+3) \\ & \ge (2p+r-k-t-2)(p+2k+t-r-1). \qquad (1)\end{aligned}$$

In view of the restrictions on r, k, t and p, the following values are possible.

For $r = 0$: $k = 0$, $t = 0$ and $p = 4$.

For $r = 1$: $k = 0$, $t = 1$ and $p = 5$, or $k = 0$, $t = 0$ and $p \in \{5, 6\}$.

For $r = 2$: $k = 0$, $t = 1$ and $p = 6$, or $k = 0$, $t = 0$ and $p \in \{6, 7, 8\}$.

For $r = 3$: $k = 0$, $t = 2$ and $p = 6$, or $k = 0$, $t = 1$ and $p \in \{6, 7, 8\}$, or $k = 1$, $t = 1$ and $p = 6$, or $k = 0$, $t = 0$ and $p \in \{6, 7, 8, 9\}$, or $k = 1$, $t = 0$ and $p \in \{6, 7\}$.

If each vertex outside of R_1 has at most $r-k-t+2$ neighbors in R_1 and each vertex in R_1 has at most $r-k-t+1$ neighbors in R_1, inequality (1) can be sharpened to

$$(2p+k+t)(r-k-t+2) \ge (2p+r-k-t-2)(p+2k+t-r). \qquad (2)$$

Hence, the following possibilities remain. For $r = 2$, $k = t = 0$ and $p = 6$, inequality (2) is valid if and only if every vertex outside of R_1 has exactly four neighbors in R_1 (except u and v) and $G[R_1]$ is 3-regular. For $r = 3$, $k = 0$, $t = 1$, inequality (2) implies that $p = 6$, and for $r = 3$ and $k = t = 0$, inequality (2) implies that $p = 6$ or 7.

In the following we distinguish the four cases $r = 0, 1, 2$ and $r = 3$.

Case 1. Assume that $p = 4$, $n = 16$, $\delta = 5$, and $t = 0$. This implies $|R_1| = 6$ and $|N[x] \cap R_1| \le 3$ for every vertex x. Let $N(u) - N(v) = \{u_1, u_2, u_3\}$ and $N(v) - N(u) = \{v_1, v_2, v_3\}$. It follows that $|N(v_i) \cap R_1| \le 2$ for $i = 1, 2, 3$. By symmetry, $|N(u_j) \cap R_1| \le 2$ for $j = 1, 2, 3$. Therefore $30 - 2m(G[R_1]) \le m(R_1, \overline{R}_1) = m(\overline{R}_1, R_1) \le 18$ which implies that $m(G[R_1]) \ge 6$. Since $\Delta(G[R_1]) \le 2$, the graph $G[R_1]$ is either a cycle of length six or consists of two disjoint cycles of length three. It follows that $\gamma(G[R_1]) \leq 2$ and hence, $\gamma(G) \leq 4$.

Case 2. Assume that $n = 4p + 1$ with $p \geq 5$. This implies that $|R| = 3p - 1$.

Subcase 2.1. Assume that $p = 5$, $n = 21$, $\delta = 6$, and $t = 1$. Then $|R_1| = 8$ and $|N[x] \cap R_1| \leq 3$ for every vertex x. Furthermore, there exists a vertex w with $|N[w] \cap R_1| = 3$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 5$. If there exists a vertex x with $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 5$. If $|N[x] \cap R_2| \leq 2$ for every vertex x, then $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| + 1 = 26$ and $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 26$. Hence $G[R_2]$ consists of five vertices and two non-incident edges, and every vertex outside of R_2 has exactly two neighbors in R_2 (except u, v and w). Therefore $R_1 - R_2$ dominates R_1 which implies that $\gamma(G) \leq 5 = p$.

Subcase 2.2. Assume that $\delta = p + 1$, $t = 0$ and $p = 5$ or 6. Then $|R_1| = 2p - 1$ and $|N[x] \cap R_1| \leq 4$ for every vertex x. Furthermore, there exists a vertex w with $|N[w] \cap R_1| = 4$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 2p - 5$. If there exists a vertex x with $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$. Otherwise, $|N[x] \cap R_2| \leq 2$ for every vertex x. We conclude that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}| - 3) = 2(2p + 3)$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| + 1 = p(2p - 5) + 1$, a contradiction for $p = 6$.

In the case $p = 5$, the last inequality is valid precisely if $m(\overline{R}_2, R_2) = 26$. Then $G[R_2]$ consists of five vertices and two non-incident edges, and every vertex outside of R_2 has exactly two neighbors in R_2 (except u, v and w).

Subcase 2.2.1. Assume that $w \in R_1$ and let $N(w) \cap R_1 = \{b_1, b_2, b_3\}$. It follows that $G[R_1]$ consists of a claw, and $|N(b_i) \cap R_2| = 2$ for $1 \leq i \leq 3$. By symmetry, $G[R_1 - N[b_i]]$ has the same structure as $G[R_1 - N[w]]$ for $i = 1, 2, 3$. It is easy to check that this yields a contradiction.

Subcase 2.2.2. Assume that w cannot be chosen such that $w \in R_1$. Then $w \notin R_1$. Let $N(w) \cap R_1 = \{a_1, a_2, a_3, a_4\}$. It follows that $|N(a_i) \cap R_2| = 2$ for $1 \leq i \leq 4$. Hence R_2 contains a vertex of degree at least three in the subgraph $G[R_1]$, a contradiction to the assumption $\Delta(G[R_1]) \leq 2$.

Case 3. Assume that $n = 4p+2$ with $p \geq 6$. This implies that $|R| = 3p$.

Subcase 3.1. Assume that $p = 6$, $n = 26$, $\delta = 7$, and $t = 1$. Then $|R_1| = 11$ and $|N[x] \cap R_1| \leq 4$ for every vertex x. Furthermore, there exits a vertex w with $|N[y] \cap R_1| = 4$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 7$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{4}{2} = 6 = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for every vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 32$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = 42$, a contradiction.

Subcase 3.2. Assume that $\delta = p + 1$, $t = 0$ and $p = 6$, 7 or 8. Then $|R_1| = 2p$ and $|N[x] \cap R_1| \leq 5$ for every vertex x.

If there exists a vertex w with $|N[w] \cap R_1| = 5$, let $R_2 = R_1 - N[w]$

with $|R_2| = 2p - 5$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for each vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 2(2p+4)$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = (2p-5)p$, a contradiction to $p \geq 6$.

There remains the case $p = 6$, every vertex outside of R_1 has exactly four neighbors in R_1 (except u and v), and $G[R_1]$ is a 3-regular graph of of order 12. Hence $G[R_1]$ contains two distinct vertices y and z such that $N_{G[R_1]}[y] \cap N_{G[R_1]}[z] = \emptyset$. Consequently, u, v, y and z dominate 22 vertices, and the remaining 4 vertices can be dominated by two further vertices. Thus $\gamma(G) \leq 6 = p$, and Case 3 is proved.

Case 4. Assume that $n = 4p + 3$ with $p \geq 6$. This implies that $|R| = 3p + 1$.

Subcase 4.1. Assume that $p = 6$, $n = 27$, $\delta = 7$, and $t = 2$. Then $|R_1| = 11$ and $|N[x] \cap R_1| \leq 4$ for every vertex x. Furthermore, there exists a vertex w such that $|N[w] \cap R_1| = 4$ Let $R_2 = R_1 - N[w]$ with $|R_2| = 7$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{4}{2} = 6 = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for each vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 34$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = 42$, a contradiction.

Subcase 4.2. Assume that $\delta = p + 1$, $t = 1$ and $p = 6, 7$ or 8. Then $|R_1| = 2p$ and $|N[x] \cap R_1| \leq 5$ for each vertex x.

If there is a vertex w with $|N[w] \cap R_1| = 5$, then we define $R_2 = R_1 - N[w]$. It follows that $|R_2| = 2p - 5$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for each vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 2(2p+5)$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = p(2p-5)$, a contradiction to $p \geq 6$.

It remains the case that $|N[x] \cap R_1| \leq 4$ for every vertex x. It follows that $p = 6$ and $2m(G[R_1]) \geq \delta|R_1| - m(\overline{R}_1, R_1) \geq 32$. Hence there exists two vertices y_1, y_2 such that $|N[y_i] \cap R_1| \geq 4$ and $N[y_1] \cap N[y_2] \cap R_1 = \emptyset$. Therefore $\{u, v, y_1, y_2\}$ dominate 23 vertices and the remaining four vertices can be dominated with two further vertices. It follows that $\gamma(G) \leq 6 = p$.

Subcase 4.3. Assume that $p = 6$, $n = 27$, $\delta = 8$, and $t = 1$. Then $|R_1| = 11$ and $|N[x] \cap R_1| \leq 4$ for every vertex x. Furthermore, there exists a vertex w with $|N[w] \cap R_1| = 4$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 7$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{4}{2} = 6 = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for every vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 34$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = 42$, a contradiction.

Subcase 4.4. Assume that $\delta = p + 2$, $t = 0$ and $p = 6$ or 7. Then $|R_1| = 2p$ and $|N[x] \cap R_1| \leq 5$ for every vertex x. Furthermore, there exists a vertex w with $|N[w] \cap R_1| = 5$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 2p - 5$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$.

In the remaining case that $|N[x] \cap R_2| \leq 2$ for every vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}_2| - 3) = 2(2p+5)$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = (p+1)(2p-5)$, a contradiction to $p \geq 6$.

Subcase 4.5. Assume that $\delta = p+1$, $t = 0$ and $6 \leq p \leq 9$. Then $|R_1| = 2p+1$ and $|N[x] \cap R_1| \leq 6$ for every vertex x.

Subcase 4.5.1. Assume that there is a vertex w with $|N[w] \cap R_1| = 6$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 2p-5$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 3$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$. In the remaining case that $|N[x] \cap R_2| \leq 2$ for each vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 2(|\overline{R}| - 3) = 2(2p+5)$ and $m(R_2, \overline{R}_2) \geq (\delta - 1)|R_2| = p(2p-5)$, a contradiction to $p \geq 6$.

Subcase 4.5.2. Assume that there is a vertex w such that $|N[w] \cap R_1| = 5$. Let $R_2 = R_1 - N[w]$ with $|R_2| = 2p-4$. If there exists a vertex x such that $|N[x] \cap R_2| \geq 4$, then $\gamma(G) \leq 4 + \frac{2p-8}{2} = p$. In the remaining case that $|N[x] \cap R_2| \leq 3$ for each vertex x, we deduce that $m(\overline{R}_2, R_2) \leq 3(|\overline{R}_2| - 3) = 3(2p+4)$ and $m(R_2, \overline{R}_2) \geq (\delta - 2)|R_2| = (p-1)(2p-4)$. This implies $p = 6$. Note that $d(u) = d(w) = \delta = p+1 = 7$. Furthermore, note that if x is a vertex of minimum degree, then $|N(x') \cap N(x)| \geq 2$ for every vertex $x' \notin N[x]$, since otherwise $\gamma(G) \leq 6$ by one of the previous cases. If there is an edge in $G[N(u)]$ or $G[N(v)]$, we deduce $\gamma(G) \leq \delta - 1 = 6$ by Theorem 4. So assume that $N(u)$ and $N(v)$ are independent. Let $N(u) \cap N(v) = \{z_1, z_2\}$. Then $|N(z_1) \cap R_1| = 5$ and $d(z_1) = 7 = \delta$. Let $D = (N(z_1) - \{u, v\}) \cup \{z_2\}$. Recall that $|N(x') \cap N(z_1)| \geq 2$ for every vertex $x' \notin N[z_1]$ by the observation above. Since there is no edge between $N(v) - N(u)$ and $\{u, z_1, z_2\}$, each of the vertices of $N(v) - N(u)$ has a neighbor in D. Analogously, each vertex of $N(u) - N(v)$ has a neighbor in D. Additionally, none of the vertices of $R_1 - N(z_1)$ is adjacent to u or v. Hence each of them has two neighbors in $N(z_1) \cap R_1$. Finally, u and v are adjacent to z_2 and z_1 is adjacent to every vertex of D but z_2. It follows that $(N(z_1) - \{u, v\}) \cup \{z_2\}$ is a dominating set of G of size $p = 6$.

Subcase 4.5.3. Assume that $|N[x] \cap R_1| \leq 4$ for each vertex x. We deduce that $m(\overline{R}_1, R_1) \leq 4(|\overline{R}_1| - 2) = 8p$ and $m(R_1, \overline{R}_1) \geq (\delta - 3)|R_1| = (p-2)(2p+1)$, a contradiction to $p \geq 6$. $\square$

For graphs G with order $n \in \{4, 5, \ldots, 15, 18, 22\}$, we give the following examples to show that the bound in Theorem 3 cannot be improved as in Theorem 5.

Example 6. *For $n = 4$ and $n = 5$ the cycles C_4 and C_5 are examples. For $n = 6$ the complete graph K_6 minus a perfect matching is such an example. For $n = 7$ the complete graph K_7 minus an almost perfect matching minus a further edge that is incident to exactly one edge of the matching is an example.*

For $n = 8$*,* $n = 9$ *and* $n = 11$ *the graphs* F_1, F_2, F_3, F_4, F_5 *in Figure 3,* F_6, F_7, F_8 *in Figure 4 and* F_9 *in Figure 5 in [14] are examples.*

The Petersen graph is such an example for $n = 10$*.*

For $n = 12, 13, 14, 15, 18, 22$ *the graphs in Figure 1 are examples. (These examples have been constructed by hand, and* $\gamma = \lfloor n/4 \rfloor + 1$ *has been verified with a basic* `C` *algorithm on a personal computer.)*

We pose the remaining cases as open problems.

Problem 7. *Let* $n \in \{17, 19, 23\}$*. Does there exist a graph* G *of order* n *and diameter two such that* $\gamma(G) = \lfloor n/4 \rfloor + 1$*?*

3 Applications of the main result

Using the bound $\gamma(G) \leq \lfloor 2n/5 \rfloor$ of McCuaig and Shepherd [8] and further results, Dunbar, Haynes and Hedetniemi [3] presented the following Nordhaus-Gaddum type results. In the following $\mathcal{B}$ is a collection of seven graphs of order at most seven.

Theorem 8 (Dunbar, Haynes & Hedetniemi [3] 2005)**.** *If* G *and* $\overline{G}$ *are connected graphs of order* n *with* $\delta(G), \delta(\overline{G}) \geq 2$ *and* $G, \overline{G} \notin \mathcal{B} \cup \{K_3 \times K_3\}$*, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \lfloor 2n/5 \rfloor + 2.$$

Theorem 9 (Dunbar, Haynes & Hedetniemi [3] 2005)**.** *If* G *and* $\overline{G}$ *are connected graphs of order* $n \geq 23$ *with* $\delta(G), \delta(\overline{G}) \geq 2$*, then*

$$\gamma(G) + \gamma(\overline{G}) = \lfloor 2n/5 \rfloor + 2$$

if and only if $\{\gamma(G), \gamma(\overline{G})\} = \{2, \lfloor 2n/5 \rfloor\}$*.*

In addition Dunbar, Haynes and Hedetniemi [3] noted that Theorem 8 holds for $n \leq 9$ if $G, \overline{G} \notin \mathcal{B} \cup \{K_3 \times K_3\}$. Furthermore, they have been able to prove that it is true for graphs G of order $n = 17$, where $\gamma(G) = \gamma(\overline{G}) = 4$, and for graphs of order $n \in \{13, 15, 18, 20, 21\}$. However, for graphs with order $n \in \{10, 11, 12\}$, they gave examples of graphs that do not fulfill the conclusion of Theorem 8. In [15], Volkmann showed that Theorem 8 holds for graphs of order $n = 19$ and $n = 22$. The remaining cases are $\gamma(G) = 3$, $\gamma(\overline{G}) = 5$, $n = 17$, and $\gamma(G) = 3$, $\gamma(\overline{G}) = 5$, $n = 16$, and $\gamma(G) = 3$, $\gamma(\overline{G}) = 4$, $n = 14$. Using Theorem 5, we present the solution for the case $n = 16$. Assume that G is a graph with $\gamma(G) = 3$ and $n = 16$ such that $\gamma(G) + \gamma(\overline{G}) = \lfloor 2n/5 \rfloor + 2 = 8$. This implies that $\gamma(\overline{G}) = 5$. Since $\gamma(G) = 3$, it follows that $\mathrm{diam}(\overline{G}) \leq 2$ (see [1]). Combining these

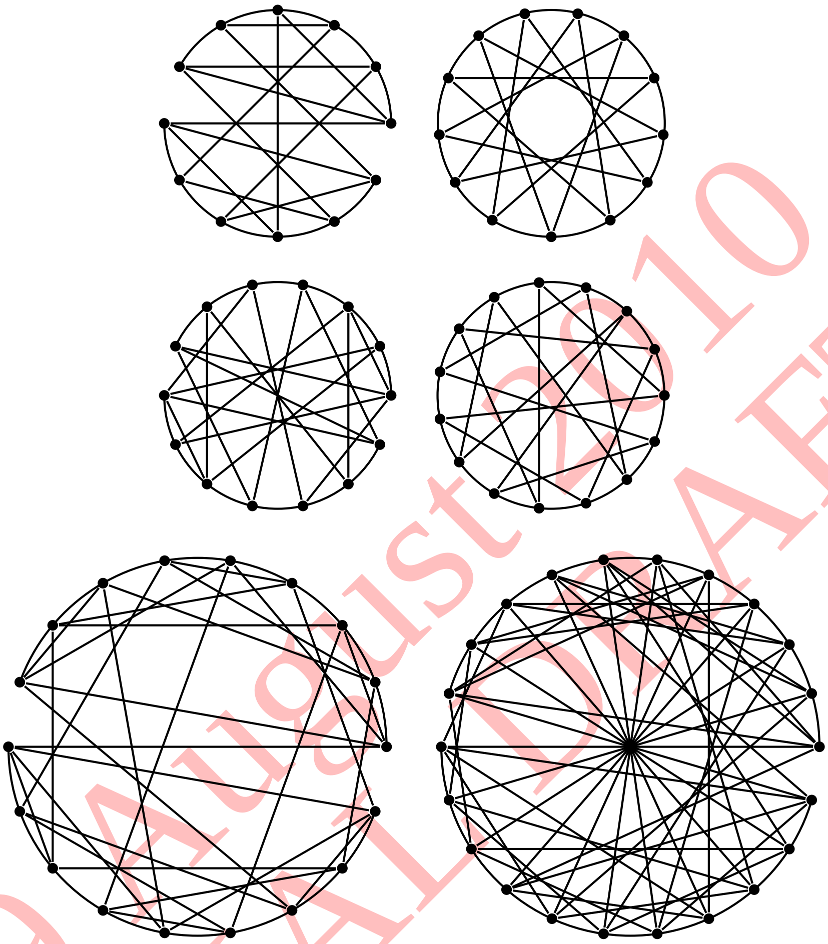

Figure 1: Graphs of order $n = 12, 13, 14, 15, 18, 22$ with diameter two and domination number equal to $\lfloor n/4 \rfloor + 1$.

observations with Theorem 5, it follows that $5 = \lfloor 2n/5 \rfloor - 1 = \gamma(\overline{G}) \leq \lfloor n/4 \rfloor = 4$, a contradiction.

Using Reed's [11] bound $\gamma(G) \leq \lfloor 3n/8 \rfloor$ for connected graphs of order n and minimum degree $\delta \geq 3$, Dunbar, Haynes and Hedetniemi [3] also established the next Nordhaus-Gaddum type result (see [15] for a new proof).

Theorem 10 (Dunbar, Haynes & Hedetniemi [3] 2005)**.** *If G and $\overline{G}$ are connected graphs of order $n \neq 10$ and $n \neq 13$ with $\delta(G), \delta(\overline{G}) \geq 3$ and $G, \overline{G} \neq K_3 \times K_3$, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \lfloor 3n/8 \rfloor + 2.$$

Analogously to Theorem 8, Volkmann investigated which graphs achieve the bound in Theorem 9.

Theorem 11 (Volkmann [15] 2010)**.** *If G and $\overline{G}$ are connected graphs with $\delta(G), \delta(\overline{G}) \geq 3$ of order $n \notin \{11, 12, \ldots, 18\} \cup \{20, 21\}$ and $G, \overline{G} \neq K_3 \times K_3$, then*

$$\gamma(G) + \gamma(\overline{G}) = \lfloor 3n/8 \rfloor + 2$$

if and only if $\{\gamma(G), \gamma(\overline{G})\} = \{\lfloor 3n/8 \rfloor, 2\}$.

In addition, for $n = 11$ and $n = 12$ Volkmann gave examples of graphs that do not fulfill the conclusion of Theorem 10. The remaining cases are $n \in \{14, 15, 16, 17, 18, 20, 21\}$. Using Theorem 5 and the following result, we present solutions for the cases $n = 16$, $n = 20$ and $n = 21$.

Theorem 12 (Jaeger & Payan [7] 1972)**.** *1. For a graph G on n vertices, $\gamma(G)\gamma(\overline{G}) \leq n$.*
2. If moreover $\gamma(G), \gamma(\overline{G}) \geq 3$, then $\gamma(G)\gamma(\overline{G}) + (\gamma(G) - 3)(\gamma(\overline{G}) - 3) \leq n$.

Assume that G is a graph of order $n \in \{16, 20, 21\}$ satisfying Theorem 10. Without loss of generality, $3 \leq \gamma(G) \leq \gamma(\overline{G})$.

If $n = 16$, then Theorem 5 implies that $\gamma(\overline{G}) \leq 4$. By our assumption, it follows that $\gamma(G) = \gamma(\overline{G}) = 4$, a contradiction to Theorem 11.

If $n = 20$ or $n = 21$, then Theorem 5 implies that $\gamma(\overline{G}) \leq 5$. By our assumption, it follows that $\gamma(G) = 4$ and $\gamma(\overline{G}) = 5$, again a contradiction to Theorem 11.

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