Sufficient conditions for triangle-free graphs to be optimally restricted edge-connected

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Abstract

For a connected graph G, an edge set S is a k-restricted edge-cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. Graphs that allow k-restricted edge-cuts are called λ_k-connected. The k-edge-degree of a graph G is the minimum number of edges between a connected subgraph H of order k and its complement $G - H$. A λ_k-connected graph is called λ_k-optimal if its k-restricted edge-connectivity equals its minimum k-edge-degree and super-λ_k if every minimum k-restricted edge-cut isolates a connected subgraph of order k.

In this paper we consider the cases $k = 2$ and $k = 3$. For triangle-free graphs that are not λ_k-optimal, we establish lower bounds for the order of components left by a minimum k-restricted edge-cut in terms of the minimum k-edge-degree. Sufficient conditions for a triangle-free graph to be λ_k-optimal and super-λ_k follow.

Keywords: restricted edge-connectivity; λ_k-optimality; super-λ_k; triangle-free graph.

1 Terminology and introduction

We consider finite graphs without loops and multiple edges. For any graph G, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. We define the *order* of G by $n = n(G) = |V(G)|$ and the *size* by $m = m(G) = |E(G)|$.

If G is a graph, then the *degree* $d(v) = d_G(v)$ of a vertex v is the number of vertices of G adjacent to v. Therefore, $\delta = \delta(G) = \min\{d(v) \colon v \in V(G)\}$ denotes the *minimum degree* of G. We call the vertex set $N_G(v) = N(v)$ of all neighbors of a vertex $v \in V(G)$ the *open neighborhood* and $N_G[v] = N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. If $A \subset V(G)$, then $N(A) = \bigcup_{v \in A} N(v) \setminus A$ and $N[A] = \bigcup_{v \in A} N[v]$. Furthermore, $G[A]$ is the graph induced by A. We write $G - H = G[V(G) \setminus V(H)]$ for a subgraph H of G, and for a subset $S \subset E(G)$ of edges, $G - S$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus S$.

A graph is called *triangle-free* if it contains no cycle of length three. A vertex set $S \subset V(G)$ is called an *independent (vertex) set* if its induced subgraph contains no edges.

Networks can be conveniently modeled as graphs, and a classical measurement for the fault tolerance of a network is the edge-connectivity. In general, the larger the edge-connectivity, the more reliable is the network.

An *edge-cut* in a connected graph G is a set S of edges of G such that $G - S$ is disconnected. An edge-cut S is called a *k-restricted edge-cut* if every component of $G - S$

has at least k vertices. Assuming that G has k-restricted edge-cuts, the *k-restricted edge-connectivity* of G, denoted by $\lambda_k(G)$, is defined as the minimum cardinality over all k-restricted edge-cuts of G, i.e.

$$\lambda_k(G) = \min\{|S|\colon S \subset E(G) \text{ is a } k\text{-restricted edge-cut}\}.$$

A connected graph G is called *λ_k-connected* if $\lambda_k(G)$ exists and a k-restricted edge-cut S is called λ_k-cut if $|S| = \lambda_k(G)$. Let $[X, \overline{X}]$ denote the set of edges between a vertex set $X \subset V(G)$ and its complement $\overline{X} = V(G) \setminus X$. Then clearly every λ_k-cut is of the form $[X, \overline{X}]$ for a set $X \subset V(G)$ and the graph $G - [X, \overline{X}]$ has exactly two components.

Let $[X, \overline{X}]$ be a λ_k-cut. Then X is called a *k-fragment* of G. We denote the minimum order of a k-fragment of G by

$$r_k(G) = \min\{|X| : X \text{ is a } k\text{-fragment of } G\}.$$

Obviously, $k \leq r_k(G) \leq |V(G)|/2$. A k-fragment X is called a *k-atom* of G if $|X| = r_k(G)$. (The number $r_2(G)$ is usually denoted by $r(G)$.)

For every positive integer k, the *minimum k-edge-degree of G* is defined as

$$\xi_k(G) = \min\{|[X, \overline{X}]|\colon |X| = k \text{ and } G[X] \text{ is connected}\}.$$

The *degree of an edge uv* is defined by $d(uv) = d(u) + d(v) - 2$. With this notation $\xi(G) = \min\{d(uv)\colon uv \in E(G)\}$ for every connected graph G. A λ_k-connected graph G with $\lambda_k(G) \leq \xi_k(G)$ is said to be *optimally k-restricted edge-connected* (for short *λ_k-optimal*) if $\lambda_k(G) = \xi_k(G)$. Note that λ_1 and λ_2 denote the *edge-connectivity* and *restricted edge-connectivity*, respectively. Accordingly, ξ_1 and ξ_2 denote the *vertex degree* and the *edge degree*. (Common notations for these values are $\lambda = \lambda_1$, $\lambda' = \lambda_2$, $\delta = \xi_1$ and $\xi = \xi_2$.) For a connected graph G it is obvious that $\lambda(G) \leq \delta(G)$. Esfahanian and Hakimi [3] showed that all graphs G except stars are λ'-connected and fulfill $\lambda'(G) \leq \xi(G)$. Bonsma, Ueffing and Volkmann [2] characterized the class of graphs that are not λ_3-connected and proved that $\lambda_3(G) \leq \xi_3(G)$ is valid for all λ_3-connected graphs and showed in the same article that in the case $k \geq 4$ the inequality $\lambda_k(G) \leq \xi_k(G)$ is not valid in general. However, Zhang and Yuan [14] showed that the inequality $\lambda_k(G) \leq \xi_k(G)$ remains true if the minimum degree is large enough.

Theorem 1.1 (Zhang & Yuan [14] 2005)**.** *Let G be a connected graph of order at least $2k$ and minimum degree $\delta \geq k - 1$. Then G is λ_k-connected and $\lambda_k(G) \leq \xi_k(G)$.*

A graph G is called *super-λ_k* if every λ_k-cut isolates a connected subgraph of order k. By definition, if G is super-λ_k, then G is λ_k-optimal. However, the converse is not true. For example, a cycle of length $n \geq 2k + 2$ is λ_k-optimal but not super-λ_k.

The *restricted edge-connectivity* was first introduced and studied by Esfahanian and Hakimi [3] in 1988. It is a special case of a quite general concept of *conditional edge-connectivity* proposed by Harary [5] in 1983 as a measurement for fault tolerance of interconnection networks. The k-restricted edge-connectivity we consider in this paper was defined by Fàbrega and Fiol [4]. Restricted edge-connectivity is one of the active research fields in graph theory as can be seen by many recent publications (see for example [11], [9], [13], [1] or [7]).

The concepts of fragments and atoms play an important role in the study of connectivity properties of graphs. Regarding the 2-restricted edge-connectivity of graphs Xu and Xu [10] and Ueffing and Volkmann [8] showed the following.

Theorem 1.2 (Xu & Xu [10] 2002)**.** *A λ'-connected graph G is λ'-optimal if and only if $r(G) = 2$.*

Theorem 1.3 (Ueffing & Volkmann [8] 2003)**.** *Every λ'-connected graph G of minimum degree δ that is not λ'-optimal fulfills $r(G) \geq \max\{3, \delta\}$.*

Equivalent results for the 3-restricted edge-connectivity of graphs were given by Bonsma, Ueffing and Volkmann [2].

Theorem 1.4 (Bonsma, Ueffing & Volkmann [2] 2002)**.** *A λ_3-connected graph G is λ_3-optimal if and only if $r_3(G) = 3$.*

Theorem 1.5 (Bonsma, Ueffing & Volkmann [2] 2002)**.** *Every λ_3-connected graph G of minimum degree δ that is not λ_3-optimal fulfills $r_3(G) \geq \max\{4, \delta - 1\}$.*

Recently, Holtkamp, Meierling and Montejano [6] considered fragments of λ'-connected triangle-free graphs.

Theorem 1.6 (Holtkamp, Meierling & Montejano [6] 2011)**.** *Let G be a λ'-connected triangle-free graph with minimum degree δ and minimum edge-degree ξ. If G is not λ'-optimal, then*

$$r(G) \geq \frac{(\delta - 1)\xi + 1}{\delta} + 2.$$

An older result of Ueffing and Volkmann [8] follows immediately.

Corollary 1.7 (Ueffing & Volkmann [8] 2003)**.** *Let G be a λ'-connected triangle-free graph with minimum degree $\delta \geq 2$. If G is not λ'-optimal, then*

$$r(G) \geq \begin{cases} 2\delta - 1 & \delta \geq 3 \\ 4 & \delta = 2 \end{cases}.$$

Holtkamp, Meierling and Montejano [6] also presented examples that show the sharpness of Theorem 1.6. In this paper we consider fragments and atoms of 2- and 3-restricted edge-cuts in triangle-free graphs. More precisely, we prove lower bounds on the order of k-fragments in λ_k-connected triangle-free graphs in terms of the minimum k-edge-degree when $k = 2$ and 3. Sufficient conditions for a graph to be λ_k-optimal and super-λ_k follow.

In [12], Zhang considered λ_k-connected graphs and proved some helpful properties of their atoms and fragments.

Lemma 1.8 (Zhang [12] 2007)**.** *Let G be a λ_k-connected graph and U a λ_k-fragment of G. Let $W \subset U$ with $|W| = k$ and $G[W]$ connected. If $\lambda_k(G) < \xi_k(G)$, then there exists a vertex $v \in N(W) \cap U$ that has at most $k-1$ neighbors outside of U.*

Lemma 1.9 (Zhang [12] 2007)**.** *Let G be a λ_k-connected graph with minimum degree at least 2 and U a λ_k-atom of G. If $|U| \geq k+1$, then every vertex of U has at least two neighbors in U.*

2 A sufficient condition for a triangle-free graph to be super-λ'

Since a smallest 2-fragment of a graph contains at most half of the graph's vertices, the following sufficient criterion for the λ'-optimality of a graph follows directly from Theorem 1.6 and Theorem 1.1.

Corollary 2.1. *Let G be a connected triangle-free graph of order n with minimum degree $\delta \geq 2$ and minimum edge-degree ξ. If*

$$2(\delta-1)\xi \geq \begin{cases} \delta(n-4)-2, & n \text{ odd} \\ \delta(n-4)-1, & n \text{ even} \end{cases},$$

then G is λ'-optimal.

Proof. If G is not λ'-optimal, then

$$r(G) \geq \frac{(\delta-1)\xi + 2\delta + 1}{\delta}$$

by Theorem 1.6. Let s be a non-negative integer and $n = 2s+1$ or $n = 2s$. If n is odd, then it follows that $r(G) \geq s+1$, and if n is even, then it follows that $r(G) > s$. Both inequalities are contradictions. ☐

The next result shows that the lower bound for the minimum edge-degree in Corollary 2.1 that guarantees λ'-optimality of a triangle-free graph is in fact sufficient for the graph to be super-λ' if its minimum degree is at least 3 and its order at least 22 and odd. If the order is even, then a slightly greater bound is needed to guarantee that the graph is super-λ'.

Theorem 2.2. *Let G be a connected triangle-free graph of order $n \geq 22$ with minimum degree $\delta \geq 3$ and minimum edge-degree ξ. If*

$$2(\delta-1)\xi \geq \begin{cases} \delta(n-4)-2, & n \text{ odd} \\ \delta(n-4)+5, & n \text{ even} \end{cases},$$

then G is super-λ'.

Proof. By Corollary 2.1, $\lambda'(G) = \lambda' = \xi$. Suppose that G has a λ'-cut that does not isolate an edge. Let U be a non-trivial λ'-fragment of G and $\overline{U} = V(G) \setminus U$ such that $|U| \le |\overline{U}|$. Let $n = 2s + \varepsilon$ with $\varepsilon \in \{0, 1\}$. Then $3 \le |U| \le s$.

Let xy be an edge in U such that $|[\{x, y\}, \overline{U}]|$ is minimal among all edges in U. Let $X = (N(x) \cap U) \setminus \{y\}$ and $Y = (N(y) \cap U) \setminus \{x\}$. Since $G[U]$ is connected, the union $X \cup Y$ is not empty and, since G is triangle-free, the sets X and Y are disjoint. The choice of xy implies that $|[v, \overline{U}]| \ge |[y, \overline{U}]|$ and $|[w, \overline{U}]| \ge |[x, \overline{U}]|$ for all vertices $v \in X$ and $w \in Y$.

Assume, without loss of generality, that $|[x, \overline{U}]| = 0$. Note that $|X| \ge \delta - 1 \ge 2$. Moreover, it is $\xi \le d(xy) = |X| + |Y| + |[x, \overline{U}]| + |[y, \overline{U}]| \le |U| - 2 + |[y, \overline{U}]|$. If $|[y, \overline{U}]| = 0$, then $n/2 \ge |U| \ge \xi + 2$, a contradiction. So assume that $|[y, \overline{U}]| \ge 1$. If $|[y, \overline{U}]| \cdot |X| \ge |X| + |Y| + 1$, then

$$\xi \le d(xy) = |X| + |Y| + |[y, \overline{U}]| \le (|X| + 1)|[y, \overline{U}]| - 1 \le |[U, \overline{U}]| - 1 = \lambda' - 1,$$

a contradiction. So $|[y, \overline{U}]| \cdot |X| \le |X| + |Y|$. It follows that $|[y, \overline{U}]| \le 1 + |Y|/|X|$ and thus,

$$\xi \le d(xy) = |X| + |Y| + |[y, \overline{U}]| \le |X| + |Y| + 1 + \frac{|Y|}{|X|}. \qquad (1)$$

Consider the function $f(x, y) = 1 + x + y + y/x$ on $M = \{(x, y) \in \mathbb{R}^2 \colon x \ge \delta - 1, y \ge 0, x + y \le |U| - 2\}$. Since $\frac{\partial f}{\partial y}(x, y) = 1 + 1/x$, the function f has no extremal point in the interior of M. If $y = 0$, then $f(x, y) = x + 1$ is maximal for $x = |U| - 2$, if $x = \delta - 1$, then $f(x, y) = y + \delta + y/(\delta - 1)$ is maximal for $y = |U| - \delta - 1$ and if $x + y = |U| - 2$, then $f(x, y) = |U| - 2 + (|U| - 2)/x$ is maximal for $x = \delta - 1$. All in all

$$\xi \le \max\{f(x, y) \colon (x, y) \in M\} = f(\delta - 1, |U| - \delta - 1) = \frac{\delta}{\delta - 1}(|U| - 2). \qquad (2)$$

If $n = 2s + 1$, then $2(\delta - 1)\xi \ge \delta(2s - 3) - 2$ and $|U| \le s$. Hence, by (2) it follows that $\delta \le 2$, a contradiction. If $n = 2s$, then $2(\delta - 1)\xi \ge \delta(2s - 4) + 5$ and $|U| \le s$, again a contradiction.

So assume that $|[x, \overline{U}]| \ge 1$ and $|[y, \overline{U}]| \ge 1$. Then every vertex of $X \cup Y$ has at least one neighbor in $\overline{U}$ and thus,

$$\xi \le d(xy) = |X| + |Y| + |[x, \overline{U}]| + |[y, \overline{U}]| \le |[X \cup Y \cup \{x, y\}, \overline{U}]| \le |[U, \overline{U}]| = \lambda'. \qquad (3)$$

Since $\lambda' = \xi$, every inequality in (3) is fulfilled with equality. In particular, $d(xy) = \xi$, $|[v, \overline{U}]| = 1$ and $|[w, \overline{U}]| = 0$ for every vertex $v \in X \cup Y$ and $w \in U \setminus N[\{x, y\}]$. If $X \cup Y \cup \{x, y\} \ne U$, then there exists a vertex $w \in U$ that is neither adjacent to x nor y, but has a neighbor in $v \in X \cup Y$, since $G[U]$ is connected. But then $|[\{vw\}, \overline{U}]| = 1$, a contradiction to the choice of xy. So $U = X \cup Y \cup \{x, y\}$. Since $\delta \ge 3$, the graph induced by $X \cup Y$ has no isolated vertices. In particular, $X \ne \emptyset$ and $Y \ne \emptyset$. Since $|[\{v, w\}, \overline{U}]| = 2$ for every edge vw between X and Y, we conclude $|[x, \overline{U}]| = |[y, \overline{U}]| = 1$ by the choice of xy. Hence, $\xi = \lambda' = |U|$ and by symmetry of its edges $G[U]$ is a complete bipartite graph.

Furthermore, $\delta - 1 \le |U_1| \le s/2$, where U_1 and U_2 are the partite sets of $G[U]$ such that $|U_1| \le |U_2|$. If $n = 2s+1$, then

$$2(\delta-1)s \ge 2(\delta-1)|U| = 2(\delta-1)\xi \ge \delta(2s-3) - 2 \ge 2s\delta - 3|U_1| - 5 \qquad (4)$$

and thus, $s \le 10$, a contradiction to $n \ge 22$. If $n = 2s$, then

$$2(\delta-1)s \ge 2(\delta-1)|U| = 2(\delta-1)\xi \ge \delta(2s-4) + 5 \ge 2(\delta-1)s + 1, \qquad (5)$$

a contradiction. This completes the proof of the theorem. □

The bounds in Theorem 2.2 are sharp in the following sense. If n is odd, then there exist graphs with $2(\delta-1)\xi = \delta(n-4)-3$ that are not λ'-optimal and therefore not super-λ' (see [6]). If n is even, then there exist graphs with $2(\delta-1)\xi = \delta(n-4)+4$ that are not super-λ'. The graphs that consist of two copies of $K_{s,s}$ that are connected by a perfect matching have minimum degree $\delta = s+1$, minimum edge-degree $\xi = 2s$ and order $n = 4s$. Hence, $2(\delta-1)\xi = 4s^2 = \delta(n-4)+4$. Moreover, if $s \ge 2$, then the perfect matching between the two copies of $K_{s,s}$ is a non-trivial λ'-cut showing that the graph is not super-λ'. Furthermore, there exist graphs with $\delta = 2$ and $2(\delta-1)\xi > \delta(n-4)-2$ that are not super-λ'. The complete bipartite graphs $K_{2,n-2}$ have minimum degree $\delta = 2$ and minimum edge-degree $\xi = n-2$. Hence, $2(\delta-1)\xi = 2n-4 > 2n-10 = \delta(n-4)-2$. But if $n \ge 6$, then the vertex set consisting of half of the vertices of each partite set is a non-trivial λ'-cut showing that $K_{2,n-2}$ is not super-λ'. Hence, the condition $\delta \ge 3$ is necessary.

The graphs of order $n \le 21$ that fulfill the assumptions of Theorem 2.2, but not its conclusion can be determined with the help of (4). If $n \le 21$, then (4) implies $|U_1| = s/2$, $s = |U| = \xi$ and $s \in \{4,6,8,10\}$ or $|U_1| = (s-1)/2$, $s = |U| = \xi$ and $s \in \{5,7\}$. Moreover, in all cases $\delta = |U_1| + 1$.

Note that n is odd. Then $|U| = s = (n-1)/2$ and thus, G is a graph of order $n = 9$, $n = 13$, $n = 17$ or $n = 21$ such that $G[U]$ is a $K_{s/2,s/2}$ or a graph of order $n = 11$ or $n = 15$ such that $G[U]$ is a $K_{(s-1)/2,(s+1)/2}$. Note that there are exactly $|U| = |\overline{U}| - 1$ edges between U and $\overline{U}$. Hence, $2|E(G[\overline{U}])| \ge |\overline{U}|(\delta-1)+1$. Furthermore, no vertex of $\overline{U}$ has neighbors in both partite sets of U, since G is triangle-free.

If $n = 9$, then $G[\overline{U}]$ is a connected triangle-free graph of order 5 with at least 6 edges and therefore isomorphic to $K_{2,3}$. Since $\delta \ge 3$, each of the three vertices of degree 2 has one neighbor in U. The remaining edge of $[U,\overline{U}]$ is incident with an arbitrary vertex of $\overline{U}$ such that G has no triangle.

If $n = 13$, then $G[\overline{U}]$ is a connected triangle-free graph of order 7 with at least 11 edges and therefore isomorphic to $K_{3,4}$ or to $K_{3,4} - e$, where e is an arbitrary edge. If $G[\overline{U}] = K_{3,4}$, then each of the four vertices of degree 3 in $G[\overline{U}]$ has one neighbor in U, since $\delta \ge 4$. The remaining two edges of $[U,\overline{U}]$ are incident with arbitrary vertices of $\overline{U}$ such that G has no triangle. If $G[\overline{U}] = K_{3,4} - e$, then each of the four vertices of degree 3 in $G[\overline{U}]$ has one neighbor in U and the vertex of degree 2 in $G[\overline{U}]$ has two neighbors in U that are in the same partite set, since $\delta \ge 4$.

If $n = 17$, then $G[\overline{U}]$ is a connected triangle-free graph of order 9 with at least 19 edges and therefore isomorphic to $K_{4,5}$ or to $K_{4,5} - e$, where e is an arbitrary edge. If

$G[\overline{U}] = K_{4,5}$, then each of the five vertices of degree 4 in $G[\overline{U}]$ has one neighbor in U, since $\delta \geq 5$. The remaining three edges of $[U, \overline{U}]$ are incident with arbitrary vertices of $\overline{U}$ such that G has no triangle. If $G[\overline{U}] = K_{4,5} - e$, then each of the five vertices of degree 4 in $G[\overline{U}]$ has one neighbor in U and the vertex of degree 3 in $G[\overline{U}]$ has two neighbors in U that are in the same partite set, since $\delta \geq 5$. The remaining edge of $[U, \overline{U}]$ is incident with an arbitrary vertex of $\overline{U}$ such that G has no triangle.

If $n = 21$, then $G[\overline{U}]$ is a connected triangle-free graph of order 11 with at least 28 edges and therefore isomorphic to $K_{5,6}$ or to $K_{5,6} - e$ or to $K_{5,6} - \{e, e'\}$, where e and e' are arbitrary edges, or to $K_{4,7}$. If $G[\overline{U}] = K_{4,7}$, then there are at least 14 edges leading from $\overline{U}$ to U, since $\delta \geq 6$, a contradiction. If $G[\overline{U}] = K_{5,6}$, then each of the six vertices of degree 5 in $G[\overline{U}]$ has one neighbor in U, since $\delta \geq 6$. The remaining four edges of $[U, \overline{U}]$ are incident with arbitrary vertices of $\overline{U}$ such that G has no triangle. If $G[\overline{U}] = K_{5,6} - e$, then each of the six vertices of degree 5 in $G[\overline{U}]$ has one neighbor in U and the vertex of degree 4 in $G[\overline{U}]$ has two neighbors in U that are in the same partite set, since $\delta \geq 6$. The remaining two edges of $[U, \overline{U}]$ are incident with arbitrary vertices of $\overline{U}$ such that G has no triangle. If $G[\overline{U}] = K_{5,6} - \{e, e'\}$, then each vertex of degree $3 \leq d \leq 5$ in $G[\overline{U}]$ is incident to exactly $6 - d$ vertices of U such that G has no triangle.

If $n = 11$, then $G[\overline{U}]$ is a connected triangle-free graph of order 6 with at least 7 edges and therefore isomorphic to $K_{2,4}$, $K_{2,4} - e$, $K_{3,3} - \{e, e'\}$, $K_{3,3} - e$ or $K_{3,3}$, where $e \neq e'$ are arbitrary edges of the respective graphs. In each of the first three cases, it is easy to see that G has at least one pair of adjacent vertices of degree 3, since there are three vertices of degree 3 in U. But then $\xi = 4$, a contradiction. Hence, $G[\overline{U}]$ is either a $K_{3,3}$ or a $K_{3,3} - e$, where e is an arbitrary edge. If $G[\overline{U}] = K_{3,3}$, then the edges between U and $\overline{U}$ are such that there are at most 3 vertices of degree 3 in $\overline{U}$, all of them in one partite set. If $G[\overline{U}] = K_{3,3} - e$, then both vertices of degree 2 in $G[\overline{U}]$ have a neighbor in U. The remaining three edges of $[U, \overline{U}]$ are distributed such that there are no two adjacent vertices of degree 3 in $\overline{U}$. This is only possible if one partite set of $G[\overline{U}]$ consists of vertices of degree 4 in G. Furthermore, the single edge between the other partite set and $G[U]$ must not connect two vertices of degree 3 in G.

Finally, if $n = 15$, then $G[\overline{U}]$ is a connected triangle-free graph of order 8 with at least 13 edges and therefore isomorphic to $K_{3,5}$, $K_{3,5} - e$, $K_{3,5} - \{e, e'\}$, $K_{4,4} - \{e, e', e''\}$, $K_{4,4} - \{e, e'\}$, $K_{4,4} - e$ or $K_{4,4}$, where e, e', e'' are arbitrary edges of the respective graphs. In each of the first five cases, it is easy to see that G has at least one pair of adjacent vertices of degree 4, since there are four vertices of degree 4 in U. But then $\xi = 6$, a contradiction. Hence, $G[\overline{U}]$ is either a $K_{4,4}$ or a $K_{4,4} - e$, where e is an arbitrary edge. If $G[\overline{U}] = K_{4,4}$, then the edges between U and $\overline{U}$ are such that there are at most 4 vertices of degree 4 in $\overline{U}$, all of them in one partite set. If $G[\overline{U}] = K_{4,4} - e$, then both vertices of degree 3 in $G[\overline{U}]$ have a neighbor in U. The remaining five edges of $[U, \overline{U}]$ are distributed such that there are no two adjacent vertices of degree 4 in $\overline{U}$.

3 A lower bound for the order of atoms in triangle-free graphs that are not λ_3-optimal

In this section we shall prove the following lower bound on the atom size of λ_3-connected triangle-free graphs.

Theorem 3.1. *Let G be a connected triangle-free graph with minimum degree at least 3. If G is not λ_3-optimal, then*

$$r_3(G) \geq \frac{\xi_3(G)}{2}.$$

Proof. By Theorem 1.1, G is λ_3-connected. Let U be a λ_3-atom, $\overline{U} = V(G) \setminus U$ and $S = [U, \overline{U}]$. By Theorem 1.4 it follows that $|U| \geq 4$. Let $U^* \subset U$ be the subset of vertices of U that are adjacent to at most two vertices outside of U. The set U^* is not empty by Lemma 1.8. We shall distinguish three cases depending on the order of the components of $G[U^*]$.

Case 1: Suppose that $G[U^*]$ has a component H of order at least 3. Let W be the vertex set of a connected subgraph of H of order 3. It follows that

$$\xi_3(G) \leq |[W, \overline{W}]| = \sum_{w \in W} |N(w) \cap \overline{U}| + \sum_{w \in W} |N(w) \cap (U \setminus W)| \leq 6 + 2(|U| - 3) = 2|U| = 2r_3(G)$$

and the desired inequality follows.

Case 2: Suppose that $G[U^*]$ does not have a component of order at least 3, but a component H of order 2. Let $V(H) = \{u, v\}$ and w a neighbor of $\{u, v\}$ with $|N(w) \cap \overline{U}| = \min\{|N(x) \cap \overline{U}| : x \in N(\{u, v\}) \cap U\}$. We define

$$\begin{aligned} A_1 &= U \cap (N(w) \setminus N[\{u, v\}]), \\ A_2 &= U \cap (N(\{u, v\}) \setminus N[w]), \\ A_3 &= U \cap (N(\{u, v, w\}) \setminus (A_1 \cup A_2)). \end{aligned}$$

(The set A_1 contains all vertices of U that are adjacent to w, but neither to u nor v; the set A_2 contains all vertices of U that are adjacent to either u or v, but not to w; and the set A_3 contains all vertices of U that are adjacent to w and to u or v.) Furthermore, let $W = \{u, v, w\}$. With this notation it is

$$\xi_3(G) \leq |[W, \overline{W}]| = |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + |N(w) \cap \overline{U}| + |A_1| + |A_2| + 2|A_3|. \quad (6)$$

Since $|N(x) \cap \overline{U}| \geq |N(w) \cap \overline{U}|$ for every vertex $x \in A_2 \cup A_3$, we conclude

$$|S| \geq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + (|A_2| + |A_3| + 1)|N(w) \cap \overline{U}|. \quad (7)$$

Combining (6) and (7) with the assumption $|S| < \xi_3(G)$, it follows that

$$(|A_2| + |A_3|)|N(w) \cap \overline{U}| \leq |A_1| + |A_2| + 2|A_3| - 1. \quad (8)$$

By Lemma 1.9, it is $A_2 \cup A_3 \neq \emptyset$. Using $|N(w) \cap \overline{U}| \geq 3$ and (8), it follows that $|A_1| - |A_2| - 1 \geq |A_2| + |A_3| \geq 1$. Now (8) implies that

$$|N(w) \cap \overline{U}| \leq \frac{|A_1| + |A_2| + 2|A_3| - 1}{|A_2| + |A_3|} = 2 + \frac{|A_1| - |A_2| - 1}{|A_2| + |A_3|} \leq |A_1| - |A_2| + 1. \quad (9)$$

Since $r_3(G) = |U| \geq 3 + |A_1| + |A_2| + |A_3|$, $|N(u) \cap \overline{U}|, |N(v) \cap \overline{U}| \leq 2$, it follows from (6) and (9) that

$$\begin{aligned} \xi_3(G) &\leq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + |N(w) \cap \overline{U}| + |A_1| + |A_2| + 2|A_3| \\ &\leq 2|A_1| + 2|A_3| + 5 \leq 2r_3(G) - 1 \end{aligned}$$

and the desired inequality is immediate.

Case 3: Suppose that U^* is an independent set. Let $X = N(U^*) \cap U = \{x_1, x_2, \ldots, x_p\}$ with $|N(x_i) \cap \overline{U}| \leq |N(x_j) \cap \overline{U}|$ for $i \leq j$.

We shall show that there exists a vertex $x \in X$ that has at least two neighbors in U^*. By Lemma 1.9, every vertex $u \in U^*$ has at least two neighbors $x, x' \in X$. Let $W = \{u, x, x'\}$. By Lemma 1.8, there exists a vertex $v \in N(W) \cap U^*$. Since U^* is independent, v is, without loss of generality, adjacent to x and thus, u and v are neighbors of x.

Now let i be minimal such that x_i has at least two neighbors in U^* and $u, v \in N(x_i) \cap U^*$ with $|N(u) \cap \overline{U}| = \min\{|N(w) \cap \overline{U}| \colon w \in N(x_i) \cap U^*\}$ and $|N(v) \cap \overline{U}| = \min\{|N(w) \cap \overline{U}| \colon w \in (N(x_i) \cap U^*) \setminus \{u\}\}$. Let $W = \{u, v, x_i\}$ and

$$\begin{aligned} A_1 &= N(\{u, v\}) \cap \{x_{i+1}, x_{i+2}, \ldots, x_p\}, \\ A_2 &= N(\{u, v\}) \cap \{x_1, x_2, \ldots, x_{i-1}\}, \\ A_3 &= (N(x_i) \cap U) \setminus (A_1 \cup A_2 \cup \{u, v\}). \end{aligned}$$

By the choice of i every vertex $w \in A_2$ is adjacent to either u or v. Hence,

$$\xi_3(G) \leq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + |N(x_i) \cap \overline{U}| + 2|A_1| + |A_2| + |A_3|. \quad (10)$$

Again by the choice of i it is $|N(w) \cap \overline{U}| \geq |N(x_i) \cap \overline{U}|$ for every vertex $w \in A_1$. Furthermore, $|N(w) \cap \overline{U}| \geq 3$ for every $w \in A_2$. It follows that

$$|S| \geq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + (|A_1| + 1)|N(x_i) \cap \overline{U}| + 3|A_2|. \quad (11)$$

Combining (10) and (11) with the assumption $|S| < \xi_3(G)$, we conclude that

$$|A_1||N(x_i) \cap \overline{U}| \leq 2|A_1| - 2|A_2| + |A_3| - 1. \quad (12)$$

If $A_1 \neq \emptyset$, then, using $|N(x_i) \cap \overline{U}| \geq 3$ and (12), we conclude that $|A_3| - 2|A_2| - 1 \geq |A_1| \geq 1$. Now (12) implies that

$$|N(x_i) \cap \overline{U}| \leq \frac{2|A_1| - 2|A_2| + |A_3| - 1}{|A_1|} = 2 + \frac{|A_3| - 2|A_2| - 1}{|A_1|} \leq |A_3| - 2|A_2| + 1. \quad (13)$$

Since $r_3(G) \geq |A_1| + |A_2| + |A_3| + 3$ and $|N(u) \cap \overline{U}|, |N(v) \cap \overline{U}| \leq 2$, it follows from (10) and (13) that

$$\begin{aligned}\xi_3(G) &\leq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + |N(x_i) \cap \overline{U}| + 2|A_1| + |A_2| + |A_3| \\ &\leq r_3(G) + |N(x_i) \cap \overline{U}| + |A_1| + 1 \\ &\leq r_3(G) + |A_1| - 2|A_2| + |A_3| + 2 \\ &\leq 2r_3(G) - 3|A_2| - 1\end{aligned}$$

and the desired inequality is immediate.

If $A_1 = \emptyset$, then we shall show that there exists a vertex $w \in A_3$ with $N(w) \cap \overline{U} = \emptyset$. Assume that $N(w) \cap \overline{U} \neq \emptyset$ for every $w \in A_3$. Recall that $|N(y) \cap \overline{U}| \geq 3$ for every vertex $y \in A_2$. Hence,

$$|S| \geq |N(u) \cap \overline{U}| + |N(v) \cap \overline{U}| + |N(x_i) \cap \overline{U}| + 3|A_2| + |A_3|,$$

a contradiction to (10) and the assumption that $|S| < \xi_3(G)$.

So there exists a vertex $w \in A_3$ with $N(w) \cap \overline{U} = \emptyset$. By the choice of u and v we conclude that $N(u) \cap \overline{U} = N(v) \cap \overline{U} = \emptyset$. Let $Y = \{u, v\} \cup \{w \in A_3 \colon N(w) \cap \overline{U} = \emptyset\}$. Then $Y \subset U^*$ and we may assume by symmetry of the vertices in Y that $N(Y) \subset \{x_1, x_2, \ldots, x_i\}$. Since $\delta(G) \geq 3$ and $N(y) \subset U$, every vertex $y \in Y$ has a neighbor in $\{x_1, x_2, \ldots, x_{i-1}\}$. Hence, for every $w \in N(W) \cap U$ one of the following cases applies:

(i) $w \notin U^*$ and $|N(w) \cap \overline{U}| \geq 3$.

(ii) $w \in U^* \setminus Y$ and $1 \leq |N(w) \cap \overline{U}| \leq 2$.

(iii) $w \in Y$ and has a neighbor $w' \in \{x_1, x_2, \ldots, x_{i-1}\}$. Note that by the choice of x_i it is $N(w_1) \cap N(w_2) = \emptyset$ for distinct vertices $w_1, w_2 \in Y$, since $N(Y) \subset \{x_1, x_2, \ldots, x_i\}$.

In case (i) there are at least three edges in S that are incident to w, but w has at most two neighbors in W and therefore adds at most two edges to $[W, \overline{W}]$. In cases (ii) and (iii) the vertex w adds at most one edge to $[W, \overline{W}]$. On the other hand, in case (ii) the vertex w is incident to at least one edge of S, and in case (iii) the vertex w' is incident to at least one edge of S. These observations lead to $|S| \geq |[W, \overline{W}]| \geq \xi_3(G)$, a contradiction to our assumption. This completes the proof of the theorem. □

The following corollary follows directly from the above theorem.

Corollary 3.2. *Every connected triangle-free graph of order n with minimum degree at least 3 and minimum 3-edge-degree $\xi_3 \geq n + 1$ is λ_3-optimal.*

Proof. If G is not λ_3-optimal, then

$$r_3(G) \geq \frac{\xi_3}{2} > \frac{n}{2}$$

by Theorem 3.1, a contradiction. □

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