Cycle factors in strongly connected local tournaments

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Abstract

A digraph without loops, multiple arcs and directed cycles of length two is called a local tournament if the set of in-neighbors as well as the set of out-neighbors of every vertex induces a tournament. A digraph is 2-connected if the removal of an arbitrary vertex results in a strongly connected digraph.

In 2004 and 2005, Li and Shu investigated the structure of strongly connected, but not 2-connected tournaments. Using their structural results they were able to give sufficient conditions for a strongly connected tournament T to have complementary cycles or a k-cycle factor, i.e. a set of k vertex disjoint cycles that span the vertex set of T.

Inspired by the articles of Li and Shu we develop in this paper the structure necessary for a strongly connected local tournament to be not cycle complementary. Using this structure, we are able to generalize and transfer various results of Li and Shu to the class of local tournaments.

Keywords: Local tournament, complementary cycles, cycle factor

1 Terminology and introduction

All digraphs mentioned in this paper are finite without loops, multiple arcs and directed cycles of length two. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

Let D be a digraph with $V(D) = \{v_1, v_2, \dots, v_r\}$ and let $H_1, H_2, \dots. H_r$ be a collection of digraphs. Then $D[H_1, H_2, \dots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and adding the arcs from every vertex of H_i to every vertex of H_j if v_iv_j is an arc of D for all i and j satisfying $1 \leq i \neq j \leq r$.

If $xy \in E(D)$, then y is a *positive neighbor* or *out-neighbor* of x and x is a *negative neighbor* or *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by

$A \rightarrow B$. Furthermore, $A \rightsquigarrow B$ denotes the fact that there is no arc leading from B to A and at least one arc is leading from A to B. In this case we also say that A *weakly dominates* B. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A, B)$ are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x)|x \in V(D)\}$ and $\min\{d^-(x)|x \in V(D)\}$, respectively. Furthermore, let $\delta(D)$ denote the minimum of $\delta^+(D)$ and $\delta^-(D)$. A digraph D with the property that $d^-(x) = d^+(x) = k$ for every vertex x of D is called k-*regular*.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \ldots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \leq i, j \leq k$, denotes the subpath $x_ix_{i+1} \ldots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then $x^+ = x_C^+$ denotes its successor on C and $x^- = x_C^-$ denotes its predecessor on C. The notations for paths are defined analogously. We call a digraph D *cycle complementary* if there exist two vertex disjoint cycles C_1 and C_2 in D such that $V(D) = V(C_1) \cup V(C_2)$. A k-*cycle factor* of a digraph D is a set of k vertex disjoint cycles which span the vertex set of D.

A digraph D is *pancyclic* if it contains cycles of length ℓ for $\ell = 3, 4, \ldots, |V(D)|$. If every vertex of D belongs to cycles of lengths $3, 4, \ldots, |V(D)|$, then D is called *vertex pancyclic*.

A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. If x is a vertex of a strongly connected digraph D such that $D - x$ is not strong, we call x a *separating vertex* or *cut-vertex* of D. Analogously, a vertex x is called a *non-separating vertex* of a strong digraph D if $D - x$ is strong. We call a digraph D k-*connected* if it has at least $k+1$ vertices and the removal of $k-1$ arbitrary vertices results in a strongly connected digraph.

A digraph is *semicomplete* if for any two distinct vertices there is at least one arc between them. An n-*tournament* is a semicomplete digraph of order n without cycles of length two. We speak of a *transitive tournament* T if there exists no cycle in T. Note that in this case T has a unique Hamiltonian path, i.e. a path that contains all vertices of T. A *local tournament* is a digraph where the inset as well as the outset of every vertex induces a tournament and a digraph is said to be *locally semicomplete* if the inset as well as the outset of every vertex induces a semicomplete digraph.

We define the tournament Q_n for $n \geq 1$ by $V(Q_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(Q_n) = \{v_iv_{i+1}|i = 1, 2, \ldots, n-1\} \cup \{v_jv_i|j \geq i+2\}$. If we consider a tournament

Q_n, we refer to the vertex v_1 as the first vertex and to the vertex v_n as the last vertex of Q_n.

Throughout this paper all subscripts are taken modulo the corresponding number.

In 1985, Reid [12] showed that every 2-connected tournament T on $n \geq 6$ vertices can be partitioned into two complementary cycles of lengths 3 and $n-3$ unless T is isomorphic to T_7^1, where T_7^1 is the tournament on seven vertices that contains no transitive subtournament on four vertices (cf. Figure 1).

Theorem 1.1 (Reid [12] 1985)**.** *Let T be a 2-connected tournament on $n \geq 6$ vertices. Then T contains two vertex disjoint cycles of lengths 3 and $n-3$, unless T is isomorphic to T_7^1.*

Using Reid's result as the induction basis, Song [13] completed this result in 1993.

Theorem 1.2 (Song [13] 1993)**.** *Let T be a 2-connected tournament on $n \geq 6$ vertices. Then T contains two vertex disjoint cycles of lengths s and $n-s$ for every s with $3 \leq s \leq \frac{n}{2}$, unless T is isomorphic to T_7^1.*

To state the next results we need some definitions. The following class of digraphs plays an important role in the study of local tournaments.

Definition 1.3. *A digraph on n vertices is called a* round digraph *if its vertices $v_1, v_2, \ldots, v_n$ can be labelled such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-d^-(v_i)}\}$ for every i, where the subscripts are taken modulo n. We refer to $v_1, v_2, \ldots, v_n$ as a* round labeling *of D.*

Definition 1.4. *A local tournament D is* round-decomposable *if there exists a round local tournament R on $r \geq 2$ vertices and strong subtournaments $H_1, H_2, \ldots, H_r$ of D such that $D = R[H_1, H_2, \ldots, H_r]$. We call $R[H_1, H_2, \ldots, H_r]$ a* round decomposition *of D.*

Definition 1.5. *Let D be a strongly connected local tournament. The* quasi-girth *$g(D)$ of D is defined as follows: If D is round-decomposable and it has the round decomposition $D = R[H_1, H_2, \ldots, H_r]$, then $g(D)$ is the length of a shortest cycle in R; if D is not round-decomposable, then $g(D) = 3$.*

Definition 1.6. *For $n \geq 5$ we define the class $\mathcal{R}_n^2$ to consist of all round local tournaments on n vertices that are 2-regular.*

In 1994, Guo and Volkmann [6] characterized all 2-connected locally semicomplete digraphs that are not cycle complementary (cf. Figure 1).

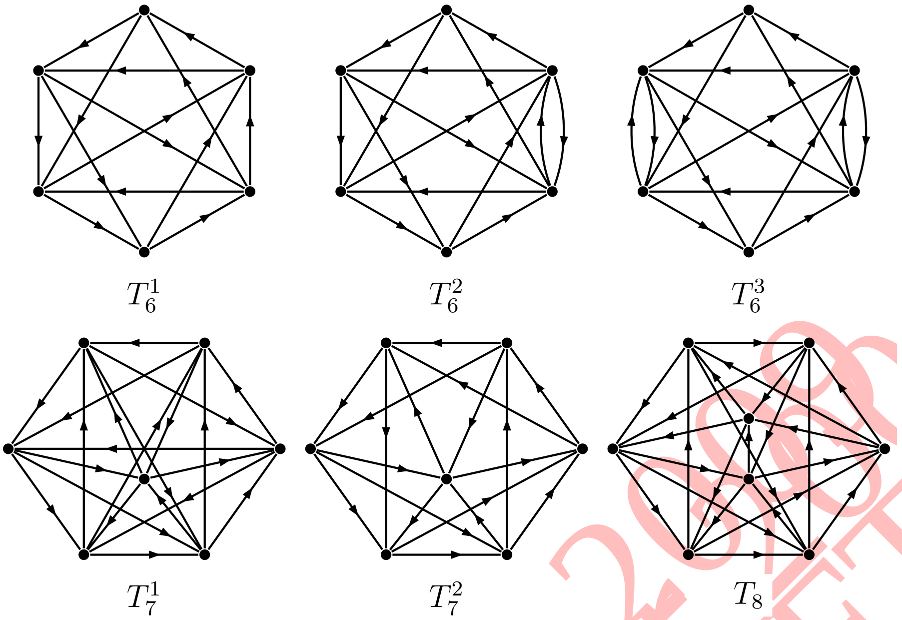

Figure 1: The locally semicomplete digraphs that are exceptions for Theorems 1.1, 1.2, 1.7 and 1.8.

Theorem 1.7 (Guo & Volkmann [6] 1994)**.** *Let D be a 2-connected locally semicomplete digraph on $n \geq 6$ vertices. Then D contains two vertex disjoint cycles that span $V(D)$, unless D is isomorphic to a member of $\{T_6^1, T_6^2, T_6^3, T_7^1, T_7^2\}$ or $\mathcal{R}_n^2$, where n is odd.*

Generalizing their own result Guo and Volkmann showed two years later that the length of the complementary cycles can be chosen somewhat arbitrary.

Theorem 1.8 (Guo & Volkmann [7] 1996)**.** *Let D be a 2-connected locally semicomplete digraph on $n \geq 6$. Then D contains two vertex disjoint cycles of lengths s and $n - s$ for every s with $g(D) \leq s \leq n - g(D)$, unless D is isomorphic to a member of $\{T_6^1, T_6^2, T_6^3, T_7^1, T_7^2, T_8\}$ or $\mathcal{R}_n^2$, where n is odd.*

Since $g(D) = 3$ if D is a tournament, the above theorem is a generalization of Theorem 1.1.

In 2005, Li and Shu [10] studied the structure of strongly connected tournaments that cannot be partitioned into two cycles.

Theorem 1.9 (Li & Shu [10] 2005)**.** *Let T be a strong tournament on $n \geq 6$ vertices that is not isomorphic to T_7^1 and let c be the number of separating vertices of T. If T is not cycle complementary, then*

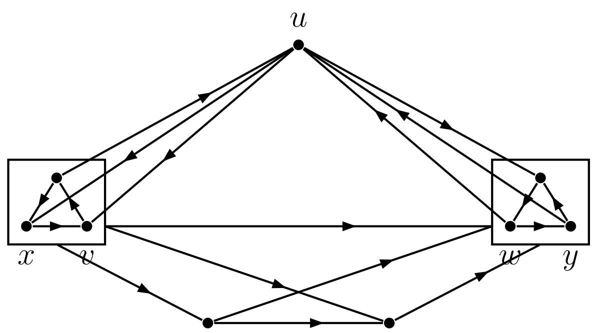

Figure 2: A counterexample for Theorem 1.9(b).

(a) $\delta^+(T) \le 2$ *and* $\delta^-(T) \le 2$;

(b) there exists *a Hamiltonian cycle* C *of* T *that can be partitioned into two consecutive segments* X *and* Y *such that* X *induces a transitive tournament,* Y *induces a tournament that is isomorphic to* $Q_{|Y|}$*, all separating vertices of* T *are consecutive on* Y *and* $|Y| = c, c+1$ *or* $c+2$.

Firstly we would like to make the following remark.

Remark 1.10. *Li and Shu stated that in the situation of Theorem 1.9(b) the Hamiltonian cycle* C *can be chosen* arbitrary*. This is not true as the tournament depicted in Figure 2 shows. The vertices* u*,* v *and* w *are the only separating vertices of* T*, but* T *has a Hamiltonian cycle* C *that starts with* ux *and ends with* yu*. So* u*,* v *and* w *are not consecutive on* C.

In particular, Li and Shu showed that a strong tournament T with $n \ge 6$ vertices that is not isomorphic to T_7^1 and has minimum out- or indegree at least three is cycle complementary.

Corollary 1.11 (Li & Shu [10] 2005)**.** *Let* T *be a strong tournament on* $n \ge 6$ *vertices such that* T *is not isomorphic to* T_7^1*. If* $\max\{\delta^-(T), \delta^+(T)\} \ge 3$*, then* T *is cycle complementary.*

Using Corollary 1.11 as the induction basis, Li and Shu [9] proved a sufficient condition for a strong tournament to have a k-cycle factor.

Theorem 1.12 (Li & Shu [9] 2004)**.** *Let* T *be a strong tournament on* n *vertices that is not isomorphic to* T_7^1 *and let* $k \ge 2$ *be a positive integer. If*

$$\delta^+(T) + \delta^-(T) \ge \frac{k-2}{k-1}n + 3k - 1,$$

then T *has a* k*-cycle factor.*

This problem — whether a strongly connected tournament can be partitioned into k cycles — was treated by Chen, Gould and Li [4] in 2001. They showed that every k-connected tournament with at least $8k$ vertices has a k-cycle factor.

Theorem 1.13 (Chen, Gould & Li [4] 2001)**.** *Let T be a k-connected tournament on $n \geq 8k$ vertices. Then T has a k-cycle factor.*

In 2004, Gould and Guo [5] discussed the same problem for the class of locally semicomplete digraphs. Generalizing the work of Chen, Gould and Li, they showed the following result.

Theorem 1.14 (Gould & Guo [5] 2004)**.** *Let D be a k-connected locally semicomplete digraph on $n \geq 5k+1$ vertices. Then D has a k-cycle factor such that one of the cycles has length at most five.*

Inspired by the articles of Li and Shu [9, 10] we develop in Section 3 the structure necessary for a strongly connected local tournament to be not cycle complementary. Using this structure, we transfer Li's and Shu's Theorem 1.9 to the class of local tournaments that are not round-decomposable (Section 4). In Section 5 we generalize Corollary 1.11 to local tournaments. The last section (Section 6) is dedicated to the existance of a k-cycle factor in strongly connected local tournaments. We show that the proposition of Theorem 1.12 is also true for local tournaments.

2 Preliminary results

The following results play an important role in our investigations.

Theorem 2.1 (Moon [11] 1966)**.** *A tournament T is strong if and only if T is vertex pancyclic.*

From this well-known result, it follows immediately that a strongly connected tournament T of order greater or equal four contains at least two non-separating vertices. This was formulated and proved by Korvin [8] in 1967.

Corollary 2.2 (Korvin [8] 1967)**.** *If T is a strong tournament with $|V(T)| \geq 4$, then T contains at least two non-separating vertices.*

The next result is a useful observation about the interaction of a cycle and an external vertex.

Lemma 2.3 (Bang-Jensen [1] 1990)**.** *Let D be a local tournament containing a cycle $C = u_1u_2 \ldots u_ku_1$. If there exists a vertex $v \in V(D) - V(C)$ such that $N^+(v, C) \neq \emptyset$ (or $N^-(C, v) \neq \emptyset$), then either $v \to C$ ($C \to v$, respectively) or $u_i \to v \to u_{i+1}$ for some integer $1 \leq i \leq k$, i.e. there exists a cycle C' in D such that $V(C') = V(C) \cup \{v\}$.*

As a consequence Bang-Jensen was able to generalize the well-known result of Camion [3] that a tournament is Hamiltonian if and only if it is strong to local tournaments.

Theorem 2.4 (Bang-Jensen [1] 1990)**.** *A local tournament is strong if and only if it has a Hamiltonian cycle.*

The next result shows that every connected local tournament has a useful decomposition.

Theorem 2.5 (Bang-Jensen [1] 1990)**.** *Let D be a connected local tournament.*

(a) If A and B are two strong components of D, then either there is no arc between them or A dominates B or B dominates A;

(b) If A and B are two strong components of D such that A dominates B, then $D[A]$ and $D[B]$ are tournaments;

(c) The strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$, where $p \geq 1$, such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.

According to Theorem 2.5, we give the following definition.

Definition 2.6. *Let D be a connected local tournament. Then the unique sequence $D_1, D_2, \ldots, D_p$ as defined in Theorem 2.5 is called the* strong decomposition *of D. Furthermore, we call D_1 the* initial strong component *and D_p the* terminal strong component *of D.*

Using the definition of a local tournament, the following property is easy to prove.

Remark 2.7. *Let D be a connected local tournament with the strong decomposition $D_1, D_2, \ldots, D_p$, where $p \geq 2$. If there exists an arc leading from D_i to D_j, where $i < j$, then $D_k \rightarrow D_\ell$ for every pair $i \leq k < \ell \leq j$ of indices.*

As the final result in this section we give the following theorem which shows that every connected local tournament belongs to one of three well-described classes.

Theorem 2.8 (Bang-Jensen, Guo, Gutin & Volkmann [2] 1997)**.** *Let D be a connected local tournament. Then exactly one of the following possibilities holds.*

(a) D is round-decomposable and there is a local tournament R on $r \geq 3$ vertices such that $R[D_1, D_2, \ldots, D_r]$ is the unique round decomposition of D, where D_i is a strong tournament for $i = 1, 2, \ldots, r$;

(b) D is not round-decomposable and not a tournament;

(c) D is a tournament that is not round-decomposable.

3 Structure of strongly connected local tournaments that are not cycle complementary

In this section we develop the structure necessary for a strongly connected local tournament to be not cycle complementary. We start with a first lemma.

Lemma 3.1. *Let D be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let u be a separating vertex of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. Then*

(a) $|V(D_i)| \leq 3$ *for every index i with $2 \leq i \leq p-1$;*

(b) *if $|V(D_i)| = 3$ for an index i with $2 \leq i \leq p-1$, then there is no arc from D_r to D_s for every pair r, s of indices with $r < i < s$;*

(c) *if $|V(D_1)| \geq 4$, then every positive neighbor of u in D_1 is a separating vertex of D_1 and if $|V(D_p)| \geq 4$, then every negative neighbor of u in D_p is a separating vertex of D_p;*

(d) *if $u \to D_1$, then $|V(D_1)| \leq 3$ and if $D_p \to u$, then $|V(D_p)| \leq 3$;*

(e) *if $|V(D_1)| \geq 3$, then u does not have any out-neighbors in D_2 and if $|V(D_p)| \geq 3$, then u does not have any in-neighbor in D_{p-1}.*

Proof. Assume that $|V(D_i)| \geq 4$ for an index i with $2 \leq i \leq p-1$. Note that D_i is a strong subtournament of D. According to Theorem 2.1, the component D_i is pancyclic and thus, contains a cycle C_i of length $|V(D_i)| - 1$. Let $x_i \in V(D_i)$ be the vertex that does not belong to C_i. Then

$$uD_1D_2 \ldots D_{i-1}x_iD_{i+1}D_{i+2} \ldots D_pu$$

and C_i are complementary cycles of D, a contradiction. So $|V(D_i)| \leq 3$ for every index i with $2 \leq i \leq p-1$ and (a) has been proved.

To prove (b) let $|V(D_i)| = 3$ for an index i with $2 \leq i \leq p-1$ and assume that there is an arc from D_r to D_s for a pair r, s of indices with $r < i < s$. In view of Remark 2.7 we obtain $D_{i-1} \to D_{i+1}$ and thus, D_i and $D - D_i$ are strong complementary subdigraphs of D, a contradiction.

Suppose now that $|V(D_1)| \geq 4$. If $u \to x \in V(D_1)$ and $D_1 - x$ is strong, then a Hamiltonian cycle of $D_1 - x$ and

$$uxD_2D_3 \ldots D_pu$$

are complementary cycles of D, a contradiction. So (c) is true.

Part (d) is a direct consequence of (c).

If $|V(D_1)| \geq 3$ and $N^+(u, D_2) \neq \emptyset$ (or if $|V(D_p)| \geq 3$ and $N^-(u, D_{p-1}) \neq \emptyset$), it is easy to see that D_1 and $D - D_1$ (D_p and $D - D_p$, respectively) are strong complementary subdigraphs of D, a contradiction. Hence (e) is true and the proof is complete. □

So if $D_p \to u \to D_1$, then D has the structural properties above. The next two lemmas cover the remaining cases.

Lemma 3.2. *Let D be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Let u be a separating vertex of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $D_p \not\to u$, then $D_p - x$ is not strong for every in-neighbor x of u in D_p. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - x$, where $q \geq 2$. Then*

(a) $D_i \to D_j$ for every pair of indices $1 \leq i < j \leq p$;

(b) $|V(D_i)| = 1$ for every index i with $2 \leq i \leq p - 1$;

(c) $V(A_i) = \{a_i\}$ for every index $i < q$;

(d) $a_i \not\to u$ for every index i with $2 \leq i \leq q - 1$;

(e) if $x \to a_2$, then $a_1 \not\to u$;

(f) if $a_{q-1} \to x$, then $|V(A_q)| = 1$;

(g) if $a_i \to x$ for an index $2 \leq i \leq q-1$, then u does not have any in-neighbors in A_q;

(h) $|N^-(u, A_q)| \leq 1$ and if $|N^-(u, A_q)| = 1$, then $N^-(x, A_q) = N^-(u, A_q)$.

Proof. Since u has out-neighbors both in D_1 and D_p, it follows by Remark 2.7 that (a) is true and thus, (b) holds in view of Lemma 3.1(b).

For the remaining part of this proof let P be a Hamiltonian path of $D - (V(D_p) \cup \{u\})$ starting in $N^+(u, D_1)$ and ending in D_{p-1}. Let x be an arbitrary in-neighbor of u in D_p. If $D_p - x$ is strong, let C_p be a Hamiltonian cycle of $D_p - x$. Then $uPxu$ and C_p are complementary cycles of D, a contradiction. Hence $D_p - x$ is not strong. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition of $D_p - x$, where $q \geq 2$. Note that $A_i \to A_j$ for every pair $1 \leq i < j \leq q$ of indices, since D_p induces a tournament in D.

If $|V(A_i)| \geq 3$ for an index $i < q$, a Hamiltonian cycle of A_i and

$$uPA_1A_2 \ldots A_{i-1}A_{i+1}A_{i+2} \ldots A_qxu$$

are complementary in D, again a contradiction. So (c) is valid.

If $a_i \to u$ for an index i with $2 \le i \le q-1$, the cycles

$$uPa_2a_3 \dots a_iu \quad \text{and} \quad xa_1a_{i+1}a_{i+2} \dots a_{q-1}A_qx$$

show that D is cycle complementary, a contradiction, and hence (d) holds.

If $x \to a_2$ and $a_1 \to u$, we obtain a contradiction by the complementary cycles

$$uPa_1u \quad \text{and} \quad xa_2a_3 \dots a_{q-1}A_qx.$$

Therefore (e) is true.

If $a_{q-1} \to x$ and $|V(A_q)| \ge 3$, let C_q be a Hamiltonian cycle of A_q. Then C_q and

$$uPa_1a_2 \dots a_{q-1}xu$$

are complementary in D, a contradiction. So (f) has been proved.

If $a_i \to x$ for an index $i < q$ and u has an in-neighbor in A_q, the cycles

$$uPa_{i+1}a_{i+2} \dots a_{q-1}A_qu \quad \text{and} \quad xa_1a_2 \dots a_ix$$

show that D is cycle complementary, again a contradiction. Hence (g) holds.

Finally, let v be an in-neighbor of x in A_q. If u has an in-neighbor $w \ne v$ in A_q, let C_q be a Hamiltonian cycle of A_q. It follows that

$$uPC_q[v^+, w]u \quad \text{and} \quad xa_1a_2 \dots a_{q-1}C_q[w^+, v]x$$

are complementary cycles in D. This final contradiction completes the proof of (h) and of this lemma. □

Lemma 3.3. *Let D be a strong local tournament on $n \ge 6$ vertices that is not cycle complementary. Let u be a separating vertex of D and let $D_1, D_2, \dots, D_p$ be the strong decomposition of $D - u$, where $p \ge 2$. If $D_p \not\to u$, then $V(D_p)$ can be partitioned in $U^+ = N^+(u, D_p) \ne \emptyset$, $U^- = N^-(u, D_p) \ne \emptyset$ and $R = V(D_p) - (U^+ \cup U^-)$ such that $|U^-| \le 2$ and*

(a) if $R \ne \emptyset$, then $U^+ \to R \to U^-$, $D[U^+]$ is a transitive tournament and $D[R]$ is a transitive tournament or a 3-cycle. Furthermore, $|U^-| = |V(D_1)| = 1$ and the unique vertex in U^- is a separating vertex of D_p and D.

(b) if $R = \emptyset$, then

- *(i) if $U^- = \{x_1, x_2\}$ such that $x_1 \to x_2$, then x_1 is the only in-neighbor of x_2 in D_p, the vertices x_1 and x_2 are consecutive on every Hamiltonian cycle of D_p and x_1 is a separating vertex of D_p and D;*
- *(ii) if $U^- = \{x_1\}$, then x_1 is a separating vertex of D_p and D.*

Proof. Note that D_p induces a strong tournament in D. Moreover, since $D_p \not\rightarrow u$, both U^+ and U^- are not empty by Lemma 2.3. Therefore u has out-neighbors both in D_1 and D_p and thus, $D_i \rightarrow D_j$ whenever $i < j$ in view of Remark 2.7. If $R \neq \emptyset$ and there exists an arc leading from U^- to R or from R to U^+, it follows that there exists an arc between u and R, a contradiction to the definition of R. So $U^+ \rightarrow R \rightarrow U^-$.

Let $U^- = \{x_1, x_2, \ldots, x_r\}$, $U^+ = \{y_1, y_2, \ldots, y_s\}$ and $R = \{z_1, z_2, \ldots, z_t\}$, where $r, s \geq 1$ and $t \geq 0$. For the remaining part of this proof let P be a Hamiltonian path of $D - (V(D_p) \cup \{u\})$ starting in $N^+(u, D_1)$ and ending in D_{p-1}. Note that $D_p - x_r$ has the structure as described in Lemma 3.2.

Firstly we shall show that $|U^-| \leq 2$. So assume that $|U^-| \geq 3$. Note that $a_i \not\rightarrow u$ for every index $2 \leq i \leq q-1$ by Lemma 3.2(d) and that u has at most one in-neighbor in A_q by Lemma 3.2(h). Hence $a_1 \rightarrow u$ and $v \rightarrow u$ for a vertex $v \in V(A_q)$. If $q \geq 3$, it follows by Lemma 3.2(g) that $x_r \rightarrow a_2$, a contradiction to Lemma 3.2(e). If $q = 2$, then A_2 contains at least three vertices, since $D_p \not\rightarrow u$. Using Lemma 3.2(h), we conclude that $|N^-(x_r, A_q)| = 1$. Hence x_r has a positive neighbor in A_q and thus, $D[V(A_2) \cup \{x_r\}]$ is strong. Then a Hamiltonian cycle of $D[V(A_2) \cup \{x_r\}]$ and uPa_1u are complementary cycles of D, a contradiction. So $|U^-| \leq 2$.

In considering the cases $R = \emptyset$ and $R \neq \emptyset$ we shall show that (a) and (b) are valid.

Case 1: Suppose that $R = \emptyset$. Recall that every vertex $x \in U^-$ is a separating vertex of D_p by Lemma 3.2.

If $U^- = \{x_1\}$, then x_1 is also a separating vertex of D, since D_p has at least three vertices.

If $U^- = \{x_1, x_2\}$ such that $x_1 \rightarrow x_2$, let $a_1, a_2, \ldots, a_{q-1}, A_q$ be the strong decomposition of $D_p - x_1$. Using Lemma 3.2(d), we conclude that either $x_2 = a_1$ or $x_2 \in V(A_q)$. But the latter assumption yields a contradiction to Lemma 3.2(h). This implies that $x_2 = a_1$ and hence, $N^-(x_2, D_p) = \{x_1\}$ and x_1 and x_2 are consecutive on every Hamiltonian cycle of D_p. All in all (b) is valid.

Case 2. Suppose that $R \neq \emptyset$.

Assume that $u \not\rightarrow D_1$. Then there exists a vertex v in D_1 that dominates u. Recall that $D_1 \rightarrow D_p$. Using the local tournament property of D, we conclude that there is an arc between u and every vertex of R, a contradiction. So $u \rightarrow D_1$ and thus $|V(D_1)| = 1$ or $|V(D_1)| = 3$ according to Lemma 3.1(d).

Assume that $|V(D_1)| = 3$. By Lemma 3.1(e) the vertex u does not have any out-neighbors in D_2. Recall that $D_2 \rightarrow D_p$. Since $u \rightarrow U^+$, there is an arc between u and every vertex of D_2. Hence $D_2 \rightarrow u$. But now $D_2 \rightarrow R$ and $D_2 \rightarrow u$ which implies the existence of an arc between u and R, a contradiction to the definition of R. Therefore the component D_1 is just a single vertex.

If $D[U^+]$ contains a cycle C, let Z_1 be a Hamiltonian path of $D[U^+ - V(C)]$, let Z_2 be a Hamiltonian path of $D[R]$ and let Z_3 be a Hamiltonian path of $D[U^-]$. Then C and $uPZ_1Z_2Z_3u$ are complementary cycles of D, a contradiction. So U^+ induces a transitive tournament in D. It follows that $D[U^+]$ has a unique Hamiltonian path, say $W_1 = y_1y_2 \dots y_s$.

If $|U^-| = 2$, let Z_2 be a Hamiltonian path of $D[R]$. Since D_p is strong, the vertex y_1 has a negative neighbor x in U^-. Furthermore, let x' be the other vertex of U^-. Then

$$y_1y_2 \dots y_sZ_2xy_1 \text{ and } uPx'u$$

are complementary cycles of D, a contradiction. Hence $|U^-| = 1$ and its unique vertex is a separating vertex of D, since $|V(D_p)| \geq 3$.

If $|R| \geq 4$ and $D[R]$ is not transitive, let C be a cycle of length less than $|R|$ in $D[R]$ and let W_2 be a Hamiltonian path of $D[R - V(C)]$. Then C and

$$uPW_1W_2x_1u$$

are complementary cycles of D, a contradiction. So $D[R]$ is a transitive tournament or a 3-cycle which proves the last part of (a). □

Note that we can derive analogous results to Lemmas 3.2 and 3.3 for the structure of D_1 if $u \not\to D_1$. Additionally we make the following observation.

Lemma 3.4. *Let D be a strong local tournament on $n \geq 6$ vertices that is not 2-connected and not cycle complementary. If D is not round-decomposable, then D has a separating vertex u such that the terminal strong component D_p of the strong decomposition $D_1, D_2, \dots, D_p$ of $D - u$, where $p \geq 2$, does not contain any separating vertices of D.*

Proof. Let u be a separating vertex of D such that the order of the terminal strong component D_p of the strong decomposition $D_1, D_2, \dots, D_p$, where $p \geq 2$, of $D - u$ is minimal. Assume that D_p contains a separating vertex v of D.

If $|V(D_p)| \geq 3$, let $A_1, A_2, \dots, A_q$ be the strong decomposition of $D_p - v$, where $q \geq 1$. Since $D - v$ is not strong, the vertex u does not have any in-neighbors in A_q. In addition, there is no arc leading from A_q to D_i for $i \leq p - 1$ and $A_j \to A_q$ for $j < q$. Hence A_q is the terminal strong component of $D - v$, a contradiction to the choice of u.

So v is the only vertex of D_p. Since v is a separating vertex of D, it follows that u does not have any in-neighbors in D_{p-1}. Since D is not round-decomposable, there is an arc from D_r to u for an index $r \leq p - 2$. Hence $D_i \to D_j$ for every pair i, j of indices with $r \leq i < j$ and $u \to D_{p-1}$. By Lemma 3.1(b), we conclude that $|V(D_{p-1})| = 1$. But then the terminal strong component of $D - v$ is D_{p-1} and the sole vertex of D_{p-1} is not a separating vertex of D. □

Analogously we can find a separating vertex v of D such that the initial strong component of the strong decomposition of $D - v$ contains no separating vertex of D.

4 The number of cut-vertices in local tournaments that are not cycle complementary

In this section we use the structural results of Section 3 to answer the question how many separating vertices can there possibly be in a local tournament that is not cycle complementary. Firstly we consider round-decomposable local tournaments.

Remark 4.1. *Let $n \geq 5$ be an odd integer. If D is a member of $\mathcal{R}_n^2$, it is a 2-connected local tournament that is not cycle complementary. Hence every subdigraph of D is not cycle complementary. It follows that for every integer c with $1 \leq c \leq n$ there exists a round local tournament D that is not cycle complementary with exactly c separating vertices which are, in general, not consecutive on a Hamiltonian cycle of D.*

In the next result we address strongly connected local tournaments that are not round-decomposable.

Theorem 4.2. *Let D be a strong local tournament that is not cycle complementary and let c be the number of separating vertices of D. If D is not round-decomposable, then D has a Hamiltonian cycle C such that all separating vertices of D are consecutive on C. In addition, if $c \geq 4$, then C can be partitioned into two consecutive segments X and Y such that*

(a) X induces a transitive tournament;

(b) Y induces a tournament that is isomorphic to $Q_{|Y|}$;

(c) all separating vertices of D are consecutive on Y;

(d) $|Y| = c, c+1$ or $c+2$.

Proof. If D is 2-connected, it has $c = 0$ separating vertices and there is nothing to show. So assume that D is strong, but not 2-connected. By the remark below Lemma 3.4, there exists a separating vertex u of D such that the strong decomposition $D_1, D_2, \ldots, D_p$ of $D - u$, where $p \geq 2$, has the property that D_1 does not contain any separating vertices of D.

Assume that $|V(D_1)| \geq 4$. Then $u \not\rightarrow D_1$ by Lemma 3.1(d). Therefore D and D_1 have the structure as described in Lemma 3.3 and thus, D_1 contains a separating vertex of D, a contradiction. So $|V(D_1)| \leq 3$.

If $|V(D_1)| = 1$, we obtain $N^+(u, D_2) \neq \emptyset$ and if $|V(D_1)| = 3$, we obtain $u \to D_1$, since D_1 does not contain any separating vertices of D. In the former case let $Y_1 = \emptyset$ and in the latter case let $Y_1 = \{w\}$, where w is an arbitrary vertex of D_1.

We consider two cases depending on the structure of D.

Case 1: Suppose that $D_p \to u$. Note that a vertex y of D_p is a separating vertex of D if and only if it is the only vertex of D_p and u does not have any in-neighbors in D_{p-1}. Since D is not round-decomposable, there exist two indices $r \leq s$ such that $N^-(u, D_r) \neq \emptyset$ and $N^+(u, D_s) \neq \emptyset$. Let r and s be chosen minimal and maximal, respectively.

If $r = s$, it follows that $2 \leq r \leq p - 1$ and $|V(D_r)| \geq 3$. By Lemma 3.1(b) we conclude that there is no arc between every pair D_i, D_j of components with $i < r < j$. Therefore $D_i \to D_j$ for every pair i, j of indices with $i < j \leq r$ or $r \leq i < j$. Now it is easy to see that u is the only separating vertex of D.

If $r < s$, it follows that $2 \leq r < s \leq p - 1$. Then $D_i \to D_j$ for every pair i, j of indices with $i < j \leq s$ or $r \leq i < j$. So D does not have any separating vertices in D_i for $i = 1, 2, \ldots, p - 1$ which implies that $c \leq 2$.

Case 2: Suppose that $D_p \not\to u$. Then D_p has the structure as described in Lemmas 3.2 and 3.3.

Subcase 2.1: Suppose that $R = \emptyset$. We shall prove the proposition by induction on $|V(D_p)|$.

Induction basis. Suppose that $|V(D_p)| = 3$. Let $C = v_1v_2v_3v_1$ be the Hamiltonian cycle of D_p. Then, without loss of generality, either $v_1 \to u \to \{v_2, v_3\}$ or $\{v_1, v_2\} \to u \to v_3$. In the former case v_1 and v_3 are the separating vertices of D in D_p and in the latter case v_1 is the separating vertex of D in D_p. So let $Y_p = \{v_1, v_3\}$ and $Y_p = \{v_1\}$, respectively. Then Y_p and $X_p = V(D_p) - Y_p$ form a partition of D_p such that X_p induces a transitive tournament, Y_p induces a tournament that is isomorphic to $Q_{|Y_p|}$, all separating vertices of D in D_p are consecutive on the Hamiltonian path from the first to the last vertex of $D[Y_p]$, $|Y_p| = c'$, where c' is the number of separating vertices of D in D_p, and the last vertex of $D[Y_p]$ dominates u. Hence $Y = Y_p \cup \{u\} \cup Y_1$ and $X = V(D) - Y$ fulfill the conditions (a)-(d).

Inductive step. Suppose that $|V(D_p)| \geq 4$. Let $x \in V(D_p)$ be an in-neighbor of u such that x is a separating vertex of D and D_p (such a vertex exists by Lemma 3.3(b)). Then $D_p - x$ has the structure described in Lemma 3.2. Let $a_1, a_2, \ldots, a_{q-1}, A_q$ be the strong decomposition of $D_p - x$, where $q \geq 2$. Then $|V(A_q)| < |V(D_p)|$. Since x is a separating vertex of D, it follows that $u \to A_q$. In addition $u \to a_i$ for every index i with $2 \leq i \leq q - 1$ by Lemma 3.2(d). Note that $D - a_i$ is strong for every index $i < q$.

If $|V(A_q)| = 1$, the only vertex a_q of A_q is a separating vertex of D if and only if $x \to a_{q-1}$. If $D - a_q$ is strong, let $Y_p = \{x\}$ and if $D - a_q$ is not strong, let $Y_p = \{a_q, x\}$.

If $|V(A_q)| = 3$ and A_q contains no separating vertex of D, let $Y_p = \{x, v\}$, where v is an arbitrary vertex of A_q.

Otherwise, by the induction hypothesis, the component A_q can be partitioned in X' and Y' such that X' induces a transitive tournament, Y' induces a tournament that is isomorphic to $Q_{|Y'|}$, all separating vertices of D in A_q are consecutive on the Hamiltonian path from the first to the last vertex of $D[Y']$, $|Y'| = c'$, where c' is the number of separating vertices of D in A_q, and the last vertex of $D[Y']$ dominates x. Let $Y_p = Y' \cup \{x\}$. Then $Y = Y_p \cup \{u\} \cup Y_1$ and $X = V(D) - Y$ fulfill the conditions (a)-(d).

Subcase 2.2: Suppose that $R \neq \emptyset$. It follows that $u \to D_i$ for $i < p$ and hence D does not have any separating vertices in D_i for $i < p$. Recall that D has the structure described in Lemma 3.3. We observe that the only vertex x_1 of U^- is a separating vertex of D. Moreover, a vertex $z \in R$ is a separating vertex of D if and only if $R = \{z\}$. Furthermore, it is easy to see that $D - y_i$ is strong for every vertex $y_i \in U^+$. Therefore $c \leq 3$ and the set of separating vertices of D induces a path of order c in D which is part of every Hamiltonian cycle of D. □

Using the proof of the above theorem, we obtain the following observations.

Remark 4.3. *In the situation of Theorem 4.2 the following propositions hold.*

(a) If $|Y| = c+2$, all vertices of $D[Y]$ are separating vertices of D except the first and last vertex and if $|Y| = c+1$, all vertices of $D[Y]$ are separating vertices of D except either the first or the last vertex;

(b) D has at most $\frac{n+1}{2}$ separating vertices;

(c) The in-degree of the first vertex of $D[X]$ and the out-degree of the last vertex of $D[X]$ is at most two.

5 Partitioning a strong local tournament into two cycles

Using the results of Section 3, we characterize all strongly connected local tournaments that are cycle complementary and have the following structure: D has a separating vertex u such that u dominates the initial strong component and is dominated by the terminal strong component of $D - u$. Note that this characterization includes the class of strongly connected, round-decomposable local tournaments that are not 2-connected.

Theorem 5.1. *Let D be a strong local tournament on $n \geq 6$ vertices that is not 2-connected. Let u be a separating vertex of D and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$, where $p \geq 2$. If $D_p \to u \to D_1$, then D is cycle complementary if and only if one of the following conditions is satisfied.*

(a) $|V(D_i)| \geq 4$ *for an arbitrary index* i*;*

(b) $|V(D_1)| = 3$ *and* u *has an out-neighbor in* D_2 *or* $|V(D_p)| = 3$ *and* u *has an in-neighbor in* D_{p-1}*;*

(c) $|V(D_i)| = 3$ *for an index* i *with* $2 \leq i \leq p-1$ *and there is an arc from* D_r *to* D_s *for a pair* r, s *of indices with* $r < i < s$.

Proof. If D satisfies one of the conditions, it is cycle complementary by Lemma 3.1. So assume that none of the conditions is satisfied. Then $|V(D_i)| \leq 3$ for all indices i and $D - D_i$ is not strong for every component D_i with $|V(D_i)| = 3$. Assume that D is cycle complementary and let C_1 and C_2 be two complementary cycles of D such that u is contained in C_1. As $D_i \to D_j$ whenever $i < j$, all vertices of C_2 are contained in one strong component D_s of $D - u$. Since C_2 has at least three vertices and $|V(D_s)| \leq 3$, it follows that $V(C_2) = V(D_s)$. Consequently $V(C_1) = V(D) - V(D_s)$, a contradiction to the assumption that $D - D_s$ is not strong. □

Using Theorem 2.8, we are now able to prove a necessary condition for a local tournament to be not cycle complementary in terms of minimal degree.

Theorem 5.2. *Let D be a strong local tournament on $n \geq 6$ vertices that is not cycle complementary. Then $\delta^+(D) \leq 2$ and $\delta^-(D) \leq 2$ unless D is isomorphic to T_7^1.*

Proof. If D is a member of $\mathcal{R}_n^2$, it is 2-regular and hence $\delta^+(D) = \delta^-(D) = 2$. So assume that D is not a member of $\mathcal{R}_n^2$. Then D has a separating vertex u by Theorem 1.7. If D is round-decomposable, it follows that $D_p \to u \to D_1$. Hence $d^+(v) \leq 2$ and $d^-(w) \leq 2$ for every vertex v of D_p and w of D_1 by Theorem 5.1. If D is not round-decomposable, it has the structure as described in Theorem 4.2. By Remark 4.3 we obtain $d^+(v) \leq 2$ for the last vertex v of X and $d^-(w) \leq 2$ for the first vertex v of X. □

The following result as well as Corollary 1.11 are immediate by Theorem 5.2.

Corollary 5.3. *Let D be a strong local tournament on $n \geq 6$ vertices that is not isomorphic to T_7^1. If $\max\{\delta^+(D), \delta^-(D)\} \geq 3$, then D is cycle complementary.*

6 Partitioning a strong local tournament into k cycles

In this section we use the structural results obtained in Sections 3 and 4 to give a sufficient criterion for a local tournament to have a k-cycle factor. The main result of Section 5 serves as our induction basis. Note that Theorem 1.12 is an immediate corollary of Theorem 6.1.

Theorem 6.1. *Let D be a strong local tournament on n vertices that is not isomorphic to T_7^1 and let $k \geq 2$ be a positive integer. If*

$$\delta^+(D) + \delta^-(D) \geq \frac{k-2}{k-1}n + 3k - 1,$$

then D has a k-cycle factor.

Proof. If $k = 2$, it follows that

$$\delta^+(D) + \delta^-(D) \geq \frac{2-2}{2-1}n + 3 \cdot 2 - 1 = 5$$

which implies that $\max\{\delta^+(D), \delta^-(D)\} \geq 3$. By Theorem 5.2 we conclude that D is cycle complementary, i.e. D has a 2-cycle factor.

Now let $k \geq 3$ be the smallest integer such that D cannot be partitioned into k cycles. Then D can be partitioned into $k - 1 \geq 2$ cycles. Let $C_1, C_2, \ldots, C_{k-1}$ be a partition of D in $k - 1$ cycles such that

A. $|V(C_1)| \geq |V(C_2)| \geq \ldots \geq |V(C_{k-1})|$;

B. $|V(C_1)|$ is maximal.

Then none of the digraphs induced by the cycles C_i is cycle complementary. Since $\delta^+(D) \leq \frac{n-1}{2}$ and $\delta^-(D) \leq \frac{n-1}{2}$, it follows that

$$n - 1 \geq \delta^+(D) + \delta^-(D) \geq \frac{k-2}{k-1}n + 3k - 1$$

which implies that

$$n \geq 3k(k-1).$$

Since $k \geq 3$, it follows particularly that $n \geq 9(k-1)$ and thus, regarding condition A,

$$|V(C_1)| \geq \frac{n}{k-1} \geq 9.$$

We partition $C_2, C_3, \ldots, C_{k-1}$ into two sets by defining

$$\mathcal{F} = \{C_i \mid i \geq 2 \text{ and } |V(C_i)| \geq 6 \text{ and } D[V(C_i)] \not\cong T_7^1\} \text{ and}$$
$$\mathcal{H} = \{C_i \mid i \geq 2 \text{ and } |V(C_i)| \leq 5 \text{ or } D[V(C_i)] \cong T_7^1\}.$$

Firstly we prove the following claim.

Claim 1. Let C_i be an arbitrary cycle, where $i \geq 2$. If v is not a separating vertex of $D[C_i]$, then either $v \to C_1$ or $C_1 \to v$ or there are no arcs between v and C_1.

Proof of Claim. Let C_i be an arbitrary cycle, where $i \geq 2$, and let v be a non-separating vertex of C_i.

If there are no arcs between v and C_1, there is nothing to show.

So suppose that v has a positive neighbor on C_1. By Lemma 2.3, either $v \to C_1$ or C_1 can be extended by v. Since the latter yields a contradiction to the choice of C_1, it follows that $v \to C_1$.

By symmetry we conclude that $C_1 \to v$ if v has a negative neighbor on C_1 and the proof of this claim is complete. □

Let u and v be two vertices of C_1 such that $d^-(u, C_1)$ and $d^+(v, C_1)$ are minimal. By Theorem 5.2, we obtain $d^-(u, C_1) \leq 2$ and $d^+(v, C_1) \leq 2$. We shall show now that

$$d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i)| + 3$$

for every $i = 2, 3, \ldots, k-1$.

Case 1: Let $C_i \in \mathcal{F}$ be an arbitrary cycle and let S_i be the set of separating vertices of $D[V(C_i)]$. Note that by Claim 1 every vertex $x \notin S_i$ of C_i contributes at most one to the sum of $d^-(u, C_i)$ and $d^+(v, C_i)$. It follows that

$$d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i) - S_i| + d^-(u, S_i) + d^+(v, S_i).$$

Thus, if $d^-(u, S_i) \leq 3$ or $d^+(v, S_i) \leq 3$ or $|S_i| \leq 3$, the desired inequality is trivially true. So assume that $|S_i| \geq 4$ and $d^-(u, S_i) + d^+(v, S_i) \geq |S_i| + 4$. Then S_i induces a tournament of order at least four in D. Since every round-decomposable local tournament has the property that every quadruple of separating vertices does not induce a tournament, it follows that C_i is not round-decomposable. Then $D[V(C_i)]$ has the structure described in Theorem 4.2. Let $P = x_1 x_2 \ldots x_s$ be the Hamiltonian path of $D[S_i]$ such that $x_q \to x_r$ for $q > r+1$ and P is a segment of C_i. Let x_0 be the predecessor of x_1 on C_i and let x_{s+1} be the successor of x_s on C_i. Since $D[V(C_i)]$ is a local tournament that is not cycle complementary, we deduce that

$\{x_4, x_5, \ldots, x_s\} \to x_0$ and $x_{s+1} \to \{x_0, x_1, \ldots, x_{s-3}\}$. Using the local tournament property of D and the fact that $d^-(u, S_i) \geq 4$ and $d^+(v, S_i) \geq 4$, we conclude that u, x_0 and v, x_{s+1} are adjacent. By Claim 1, we deduce that either $x_0 \to C_1$ or $C_1 \to x_0$ and that either $x_{s+1} \to C_1$ or $C_1 \to x_{s+1}$. Let $C_i[x_{s+1}, x_0] = y_1 y_2 \ldots y_t$.

Subcase 1.1: Suppose that $v \to x_{s+1}$ and $x_0 \to u$. Then $x_0 \to C_1 \to x_{s+1}$. It follows that

$$C = vC_i[x_{s+1}, x_0]C_1[v^+, v] \quad \text{and} \quad x_1 x_2 \ldots x_s x_1$$

are complementary cycles in $D[V(C_1) \cup V(C_i)]$ such that $|V(C)| > |V(C_1)|$, a contradiction.

Subcase 1.2: Suppose that $x_{s+1} \to v$ and $x_0 \to u$. Then $\{x_0, x_{s+1}\} \to C_1$. Furthermore, there exists a vertex x_j with $4 \leq j \leq s$ such that $v \to x_j$. But then

$$C = C_1[v^+, v]C_i[x_j, x_0]v^+ \quad \text{and} \quad x_1 x_2 \ldots x_{j-1} x_1$$

are complementary cycles in $D[V(C_1) \cup V(C_i)]$ such that $|V(C)| > |V(C_1)|$, again a contradiction.

By symmetry the case that $v \to x_{s+1}$ and $u \to x_0$ can be solved analogously.

Subcase 1.3: Suppose that $x_{s+1} \to v$ and $u \to x_0$. Then $x_{s+1} \to C_1 \to x_0$. As $C_1 \to x_0$, the digraph $D[V(C_1)]$ is a strong tournament and consequently contains a cycle C' of length three. Let $A = V(C_1) - V(C')$. Since $x_{s+1} \to C_1 \to x_0$, every vertex of A has at least one out-neighbor and at least one in-neighbor in C_i. By Lemma 2.3, it follows that $D[V(C_i) \cup A]$ has a Hamiltonian cycle C. But then C and C' are complementary cycles of $D[V(C_1) \cup V(C_i)]$ and as $|V(C_i)| \geq 4 > |V(C')|$, we have $|V(C)| > |V(C_1)|$, a contradiction.

Case 2. Let $C_i \in \mathcal{H}$ be an arbitrary cycle. Let $R_i = \{w \in V(C_i) \mid v \to w \to u\}$. We will show that $|R_i| \leq 3$.

If $|V(C_i)| = 3$, the proposition is immediate.

If $|V(C_i)| = 4$ or $|V(C_i)| = 5$, assume that $|R_i| \geq 4$. Then $d^-(u, C_i) + d^+(v, C_i) \geq |V(C_i)| + 4$. It follows that $v \to C_i$ or $C_i \to u$ and C_i induces a strong tournament T_i in D. By Corollary 2.2 there exist two non-separating vertices in T_i. But then at least one of these vertices has an in- as well as an out-neighbor on C_1, a contradiction to Claim 1. Hence $|R_i| \leq 3$.

If C_i is isomorphic to T_7^1, the tournament $D[V(C_i)]$ is 2-connected and thus, $R_i = \emptyset$.

By the cases above, it follows that $d^-(u, C_i) + d^+(v, C_i) \leq |V(C_i)| + 3$ for every index $i = 2, 3, \ldots, k-1$. Therefore

$$\begin{aligned}
\delta^+(D) + \delta^-(D) &\le 4 + \sum_{i=2}^{k-1} (|V(C_i)| + 3) \\
&= 4 + \sum_{i=2}^{k-1} |V(C_i)| + \sum_{i=2}^{k-1} 3 \\
&= 4 + (n - |V(C_1)|) + 3(k-2) \\
&\le \frac{k-2}{k-1} n + 3k - 2,
\end{aligned}$$

a contradiction to our assumption. This contradiction completes the proof of this theorem. □

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