On the number of cycles in local tournaments

Dirk Meierling and Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
e-mail: {meierling,volkm}@math2.rwth-aachen.de

Abstract

A digraph without loops, multiple arcs and directed cycles of length two is called a local tournament if the set of in-neighbors as well as the set of out-neighbors of every vertex induces a tournament. A vertex of a strongly connected digraph is called a non-separating vertex if its removal preserves the strong connectivity of the digraph in question.

In 1990, Bang-Jensen showed that a strongly connected local tournament does not have any non-separating vertices if and only if it is a directed cycle. Guo and Volkmann extended this result in 1994. They determined the strongly connected local tournament with exactly one non-separating vertex. In the first part of this paper we characterize the class of strongly connected local tournaments with exactly two non-separating vertices.

In the second part of the paper we consider the following problem: Given a strongly connected local tournament D of order n with at least $n+2$ arcs and an integer $3 \leq r \leq n$. How many directed cycles of length r exist in D? For tournaments this problem was treated by Moon in 1966 and Las Vergnas in 1975. A reformulation of the results of the first part shows that we have characterized the class of strongly connected local tournaments with exactly two directed cycles of length $n-1$. Among other things we show that D has at least $n-r+1$ directed cycles of length r for $4 \leq r \leq n-1$ unless it has a special structure. Moreover, we characterize the class of local tournaments with exactly $n-r+1$ directed cycles of length r for $4 \leq r \leq n-1$ which generalizes a result of Las Vergnas.

Keywords: local tournament; number of cycles; cut vertex

1 Terminology and introduction

All digraphs mentioned here are finite without loops, multiple arcs and directed cycles of length two. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$.

Let D be a digraph with $V(D) = \{v_1, v_2, \dots, v_r\}$ and let $H_1, H_2, \dots .H_r$ be a collection of digraphs. Then $D[H_1, H_2, \dots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and adding the arcs from every vertex of H_i to every vertex of H_j if v_iv_j is an arc of D for all i and j satisfying $1 \leq i \neq j \leq r$.

If $xy \in E(D)$, then y is a *positive neighbor* or *out-neighbor* of x and x is a *negative neighbor* or *in-neighbor* of y, and we also say that x *dominates* y and that y *is dominated by* x, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and that B *is dominated by* A, denoted by $A \to B$. Furthermore, $A \leadsto B$ denotes the fact that there is no arc leading from B to A and at least one arc is leading from A to B. In this case we also say that A *weakly dominates* B. The *outset* $N^+(x)$ of a vertex x is the set of positive neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A,B)$ is the set of vertices in B to which there is an arc from a vertex in A. The *insets* $N^-(x)$ and $N^-(A,B)$ are defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x)|x \in V(D)\}$ and $\min\{d^-(x)|x \in V(D)\}$, respectively. Furthermore, let $\delta(D)$ denote the minimum of $\delta^+(D)$ and $\delta^-(D)$.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. The length of a cycle C or a path P is the number of arcs included in C or P. Let $C = x_1x_2 \ldots x_kx_1$ be a cycle of length k. Then $C[x_i, x_j]$, where $1 \leq i, j \leq k$, denotes the subpath $x_ix_{i+1} \ldots x_j$ of C with *initial vertex* x_i and *terminal vertex* x_j. Furthermore, if x is a vertex of C, then $x^+ = x_C^+$ denotes its successor on C and $x^- = x_C^-$ denotes its predecessor on C. The notations for paths are defined analogously.

A digraph D is *vertex k-pancyclic* if every vertex belongs to a cycle of length ℓ for $\ell = k, k+1, \ldots, |V(D)|$. If $k = 3$, we call D *vertex pancyclic*.

A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y.

An *n-tournament* is an orientation of a complete undirected graph of order n. A *local tournament* is a digraph where the inset as well as the outset of every vertex induces a tournament.

Throughout this paper all subscripts are taken modulo the corresponding number.

In 1966, Moon [10] proved that in a strongly connected tournament every vertex belongs to cycles of arbitrary lengths.

Theorem 1.1 (Moon [10] 1966)**.** *A tournament T is strong if and only if T is vertex pancyclic.*

From Moon's result it follows immediately that a strongly connected tournament T of order greater or equal four contains at least two non-separating vertices.

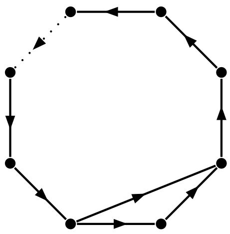

Figure 1: The local tournament with exactly one non-separating vertex.

This was formulated and proved by Korvin [6] in 1967.

Corollary 1.2 (Korvin [6] 1967)**.** *If T is a strong tournament with $|V(T)| \geq 4$, then T contains at least two non-separating vertices.*

In 1975, Las Vergnas [8] determined all strongly connected tournaments with exactly two non-separating vertices.

Theorem 1.3 (Las Vergnas [8] 1975)**.** *A strong tournament T on n vertices has at least three non-separating vertices, unless T is isomorphic to Q_n, where Q_n is the tournament of order n consisting of a path $x_1x_2\ldots x_n$ and all arcs x_ix_j for $i > j+1$.*

In 1990, Bang-Jensen [1] proved that every strongly connected locally semicomplete digraph that is not a cycle has at least one non-separating vertex.

Theorem 1.4 (Bang-Jensen [1] 1990)**.** *Let D be a strongly connected local tournament that is not a cycle. Then D has a non-separating vertex.*

Four years later Guo and Volkmann [5] showed that every strongly connected locally semicomplete digraph on $n \geq 4$ vertices has at least two non-separating vertices if it has at least $n+2$ arcs and determined the digraph in Figure 1 to be the only locally semicomplete digraph with exactly one non-separating vertex.

Theorem 1.5 (Guo & Volkmann [5] 1994)**.** *Let D be a strongly connected local tournament.*

(a) If D has at least $|V(D)|+2$ arcs, then D has at least two non-separating vertices;

(b) The digraph D has exactly one non-separating vertex if and only if D is isomorphic to the digraph depicted in Figure 1;

(c) Every vertex of D is a separating vertex if and only if D is a cycle.

In 2006, Kotani [7] investigated how many non-separating vertices a tournament with minimum degree greater or equal two has at the least. Inspired by this article, Meierling and Volkmann [9] generalized her results in considering the class of local tournaments.

Theorem 1.6 (Meierling & Volkmann [9] 2007)**.** *Let D be a strong local tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then D has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

In Section 3 we characterize all strongly connected local tournaments with exactly two non-separating vertices.

In Section 4 we further investigate the following problem.

Problem 1.7. *Given a strong local tournament D on n vertices and an integer r with $3 \leq r \leq n$. How many cycles of length r exist in D?*

In this context Moon [10] proved the following result for tournaments in 1966.

Theorem 1.8 (Moon [10] 1966)**.** *Let T be a strong tournament on n vertices and let r be an integer such that $3 \leq r \leq n$. Then T has at least $n - r + 1$ cycles of length r for every $3 \leq r \leq n$.*

In 1975, Las Vergnas [8] characterized all strongly connected tournaments with minimal number of cycles of a given length $4 \leq r \leq n - 1$.

Theorem 1.9 (Las Vergnas [8] 1975)**.** *Let T be a strong tournament on $n \geq 5$ vertices. Then T has at least $n - r + 2$ cycles of length r for every $4 \leq r \leq n - 1$ unless T is isomorphic to Q_n (cf. Theorem 1.3).*

The class of strongly connected tournaments with exactly one Hamiltonian cycle was characterized by Douglas [4] in 1970, and the class of strongly connected tournaments with exactly $n - 2$ cycles of length three was characterized by Burzio and Demaria [3] in 1990.

Bang-Jensen, Guo, Gutin and Volkmann [2] investigated which local tournaments are vertex pancyclic. In order to state their result we need the following definitions.

Definition 1.10. *A digraph on n vertices is called a* round digraph *if its vertices $v_1, v_2, \ldots, v_n$ can be labelled such that $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-d^-(v_i)}\}$ for every i, where the subscripts are taken modulo n. We refer to $v_1, v_2, \ldots, v_n$ as a* round labeling *of D.*

Definition 1.11. *A local tournament D is* round-decomposable *if there exists a round local tournament R on $r \geq 2$ vertices and strong local subtournaments $H_1, H_2, \ldots, H_r$ of D such that $D = R[H_1, H_2, \ldots, H_r]$. We call $R[H_1, H_2, \ldots, H_r]$ a* round decomposition *of D.*

Definition 1.12. *Let D be a strongly connected local tournament. The* quasi-girth *$g(D)$ of D is defined as follows: If D is round-decomposable and it has the round decomposition $D = R[H_1, H_2, \ldots, H_r]$, then $g(D)$ is the length of a shortest cycle in R; if D is not round-decomposable, then $g(D) = 3$.*

We are now able to present Bang-Jensen, Guo, Gutin and Volkmann's result.

Theorem 1.13 (Bang-Jensen, Guo, Gutin & Volkmann [2] 1997)**.** *Let D be a strongly connected local tournament on $n \geq 4$ vertices that is not a cycle.*

(a) If D is round-decomposable, then D is vertex $(g(D)+1)$-pancyclic;

(b) If D is not round-decomposable, then D is vertex pancyclic.

In Section 4 we investigate Problem 1.7 and transfer Theorems 1.8 and 1.9 to the class of local tournaments.

2 Preliminary results

Theorem 2.1 (Bang-Jensen [1] 1990)**.** *A local tournament is strong if and only if it has a Hamiltonian cycle.*

Lemma 2.2 (Bang-Jensen [1] 1990)**.** *Let D be a local tournament containing a cycle $C = u_1u_2 \ldots u_ku_1$. If there exists a vertex $v \in V(D) - V(C)$ such that $N^+(v, C) \neq \emptyset$ (or $N^-(C, v) \neq \emptyset$), then either $v \to C$ ($C \to v$, respectively) or $u_i \to v \to u_{i+1}$ for some integer $1 \leq i \leq k$, i.e. there exists a cycle C' in D such that $V(C') = V(C) \cup \{v\}$.*

Theorem 2.3 (Bang-Jensen [1] 1990)**.** *Let D be a connected local tournament.*

(a) If A and B are two strong components of D, then either there is no arc between them or A dominates B or B dominates A;

(b) If A and B are two strong components of D such that A dominates B, then $D[A]$ and $D[B]$ are tournaments;

(c) The strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$, where $p \geq 1$, such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.

According to the above theorem we give the following definition.

Definition 2.4. *Let D be a connected local tournament. Then the unique sequence $D_1, D_2, \ldots, D_p$ as defined in Theorem 2.3 is called the* strong decomposition *of D. Furthermore, we call D_1 the* initial strong component *and D_p the* terminal strong component *of D.*

From the fact that every connected non-strong local tournament has a unique strong decomposition, Guo and Volkmann [5] found a further useful decomposition, which is formulated in the next theorem.

Theorem 2.5 (Guo & Volkmann [5] 1994)**.** *Let D be a connected local tournament that is not strong. Then D can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, \ldots, D'_r$ such that every D'_i is a tournament and the following holds.*

(a) D'_1 is the terminal component of D and D'_i consists of some strong components of D for $i \geq 2$;

(b) D'_{i+1} dominates the initial strong component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \ldots, r-1$;

(c) If $r \geq 3$, then there is no arc between D'_i and D'_j for i and j satisfying $|j - i| \geq 2$.

According to Theorem 2.5, we give the following definition.

Definition 2.6. *Let D be a connected local tournament that is not strong. Then the unique sequence $D'_1, D'_2, \ldots, D'_r$ as defined in Theorem 2.5 is called the* semicomplete decomposition *of D.*

Now we give a characterization of strongly connected local tournaments that are neither tournaments nor round-decomposable.

Theorem 2.7 (Bang-Jensen, Guo, Gutin & Volkmann [2] 1997)**.** *Let D be a connected local tournament which is not a tournament. Then D is not round-decomposable if and only if the following conditions are satisfied:*

(a) There is a minimal separating set S of D such that $D - S$ is not a tournament and for each such S the digraph $D[S]$ is a tournament and the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2 and D'_3;

(b) There are integers α, β, μ and ν with $\lambda \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that

$$N^-(D_\alpha, D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha, D_\nu) \neq \emptyset$$
$$\text{or } N^-(D_\mu, D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu, D_\beta) \neq \emptyset,$$

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and D_λ is the initial component of D'_2.

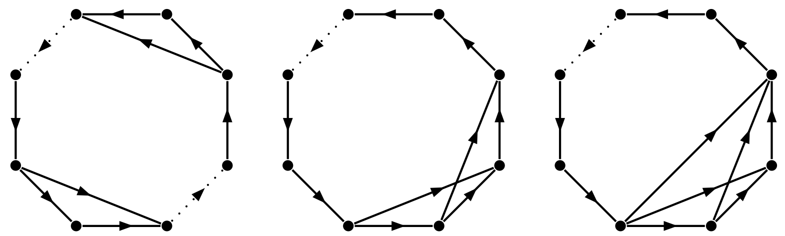

Figure 2: Local tournaments C_n^j with exactly two non-separating vertices.

3 The number of non-separating vertices in local tournaments with low degree

In [9], Meierling and Volkmann considered the following problem: how many non-separating vertices does a strongly connected local tournament have at the least if the minimum degree δ is greater or equal than $p \geq 2$? They showed that the answer is $k = \min\{|V(T)|, 4p-2\}$ (cf. Theorem 1.6). In this section we discuss the same question for strongly connected local tournaments with no restrictions on the minimum degree. We characterize all strongly connected local tournaments that have exactly two non-separating vertices thereby generalizing Theorem 1.3 for tournaments. Firstly we would like to remark the following.

Remark 3.1. *The tournament Q_n has only two non-separating vertices, the vertices x_1 and x_n. So every local tournament D on n vertices that is a subdigraph of Q_n has at most two non-separating vertices.*

3.1 The exceptional cases

Recall that a strongly connected local tournament has no non-separating vertex if and only if it is a cycle and exactly one non-separating vertex if and only if it is isomorphic to the digraph depicted in Figure 1. Therefore we concentrate on strongly connected local tournaments D that have at least $|V(D)|+2$ arcs. In this subsection we present two classes of local tournaments that have exactly two non-separating vertices (cf. Figure 2).

Definition 3.2. *Let $C = x_1x_2\ldots x_nx_1$ be a cycle on n vertices. Let*

(a) C_n^j be the strong local tournament that consists of the cycle C and the additional arcs x_1x_3 and x_jx_{j+2}, where $j \in \{2, 3, \ldots, n\}$;

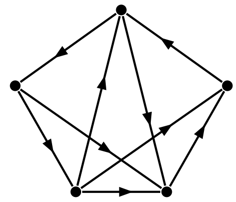

Figure 3: The local tournament Q_5^* with exactly two non-separating vertices.

(b) C_n^1 *be the strong local tournament that consists of the cycle* C *and the additional arcs* x_1x_3*,* x_1x_4 *and* x_2x_4*.*

It is easy to see that the only non-separating vertices of C_n^j (for $j = 2, 3, \dots, n$) are x_2 and x_{j+1} and the only non-separating vertices of C_n^1 are x_2 and x_3.

In the next example we present a local tournament that is not a tournament and has exactly two non-separating vertices, but is not isomorphic to C_n^j for $j = 1, 2, \dots, n$ (cf. Figure 3). It is a subdigraph of the tournament Q_5.

Definition 3.3. *Let* Q_5^* *be the local tournament* $Q_5 - x_4x_2$*, where* Q_5 *is defined as in Theorem 1.3.*

Since Q_5^* is a subdigraph of Q_5, the vertices x_1 and x_5 are the only non-separating vertices of Q_5^* (cf. Remark 3.1).

3.2 Local tournaments with exactly two non-separating vertices

In this subsection we characterize all local tournaments that have exactly two non-separating vertices to be the exceptional cases of Subsection 3.1.

Theorem 3.4. *Let* D *be a strong local tournament on* n *vertices with at least* $n + 2$ *arcs. Then* D *has exactly two non-separating vertices if and only if* D *is isomorphic to* Q_5^**,* Q_n *or* C_n^j *for an integer* $1 \le j \le n$*.*

Proof. It is easy to check that the local tournaments Q_n, Q_5^* and C_n^j for $j = 1, 2, \dots, n$ have only two non-separating vertices.

So let D be a strong local tournament on n vertices with at least $n + 2$ arcs that is neither isomorphic to Q_n nor to Q_5^* nor to C_n^j for $j = 1, 2, \dots, n$. We shall show by induction on n that D has at least three non-separating vertices. Note that $n \ge 4$, since D has at least $n + 2$ arcs.

Induction basis. The only strong local tournament on $n = 4$ vertices is isomorphic to Q_4. So the proposition is valid for $n = 4$.

Inductive step. Let $n \geq 5$ be an integer. Let D be a strong local tournament on n vertices with at least $n+2$ arcs that is neither isomorphic to Q_n nor to Q_5^* nor to C_n^j for $j = 1, 2, \ldots, n$. Observe that D has at least $n+3$ arcs, since D is not isomorphic to C_n^j for $j = 2, 3, \ldots, n$. In addition, since D is not isomorphic to C_n^1, it follows that D has at least three vertices with out-degree at least two.

If D is 2-connected, the local tournament D does not have any separating vertices. So assume that D has a separating vertex s and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - s$, where $p \geq 2$. Note that, according to Theorem 2.3, the vertex s has an out-neighbor in D_1 and an in-neighbor in D_p. By Lemma 2.2, it follows that either $s \rightarrow D_1$ or that s has positive as well as negative neighbors in D_1. Analogously, either $D_p \rightarrow s$ or s has negative as well as positive neighbors in D_p. We distinguish the following cases.

Case 1: Suppose that $D_p \rightarrow s \rightarrow D_1$.

If $D - s$ has a strong component D_i with at least three vertices, we conclude that $D - v$ is strong for every vertex v of D_i. Hence D has at least three non-separating vertices.

So assume that every strong component of $D - s$ consists of exactly one vertex. Let $V(D_i) = \{x_i\}$ for $i \in \{1, 2, \ldots, p = n-1\}$. If $d^+(x_i) \geq 2$ for an index $i < n-1$, we conclude by Theorem 2.3 that $x_i \rightarrow x_{i+2}$ (or $x_{n-2} \rightarrow s$ if $i = n-2$). Hence $D - x_{i+1}$ is strong. Since D has at least three vertices with out-degree at least two, we obtain that at least two vertices of $\{x_2, x_3, \ldots, x_{n-1}\}$ are non-separating vertices of D. Now we consider the vertex s.

If $d^+(s) = 1$, there exist three vertices x_i, x_j, x_k with out-degree at least two. Due to our observations above, x_{i+1}, x_{j+1} and x_{k+1} are non-separating vertices of D.

So assume that $d^+(s) \geq 2$. Then either $s \rightarrow x_2$ or there exists an index $2 \leq i \leq n-3$ such that $x_i \rightarrow s \rightarrow x_{i+1}$. In the first case $D - x_1$ is strong and we are done. In the second case we distinguish the cases $n = 5$ and $n \geq 6$.

If $n = 5$, we conclude that $i = 2$ and the local tournament D is either isomorphic to Q_5 or to Q_5^*, a contradiction.

So assume that $n \geq 6$. If $i = 2$, we conclude that $x_1 \rightarrow x_3$ and that $x_2 \rightarrow \{x_4, x_5, \ldots, x_{n-1}\}$. The latter implies particularly that $x_3 \rightarrow x_5$ and thus, x_2, x_3 and x_4 are non-separating vertices of D. So assume that $i \geq 3$. In this case $x_1 \rightarrow \{x_3, x_4, \ldots, x_{i+1}\}$ and $x_i \rightarrow \{x_{i+2}, x_{i+3}, \ldots, x_{n-1}\}$. The former implies particularly that $x_2 \rightarrow x_4$ and therefore x_2, x_3 and x_{i+1} are non-separating vertices of D. This completes the proof of Case 1.

Case 2: Suppose that $s \not\rightarrow D_1$ or $D_p \not\rightarrow s$. So s has in-neighbors in D_1 and D_p or out-neighbors in D_1 and D_p. It follows by Theorem 2.3 that $D_i \rightarrow D_j$ for all indices $i < j$. Hence $D - v$ is strong for every vertex v in D_i with $2 \leq i \leq p-1$.

If $|V(D_1)| \geq 3$, let C_1 be a Hamiltonian cycle of D_1 and let x_1^+ be an out-neighbor of s in D_1. Then $C_1[x_1^+, x_1^-]$ is a Hamiltonian path of $D_1 - x_1$ and, since $D_1 \to D_2$, it is easy to see that $D - x_1$ is strong. It follows that the number of non-separating vertices of D in D_1 is at least $|N^+(s, D_1)|$. Analogously we can show that if $|V(D_p)| \geq 3$, the number of non-separating vertices of D in D_p is at least $|N^-(s, D_p)|$.

If $|V(D_1)| \geq 3$ and $x \neq y$ are two vertices of D_1 such that x is an out-neighbor of s and $D_1 - y$ is strong, the vertex y is a non-separating vertex of D. Analogously if $|V(D_p)| \geq 3$ and $v \neq w$ are two vertices of D_p such that v is an in-neighbor of s and $D_p - w$ is strong, the vertex w is a non-separating vertex of D.

Since $s \not\to D_1$ or $D_p \not\to s$, we may assume, without loss of generality, that $|V(D_1)| \geq 3$. Recall that there is at least one non-separating vertex of D in D_1 and that every vertex v in D_i with $2 \leq i \leq p-1$ is a non-separating vertex. So we assume that $p \leq 3$ and $\sum_{i=2}^{p-1} |V(D_i)| \leq 1$. It remains to consider the cases $p = 3$ and $|V(D_2)| = |V(D_3)| = 1$ (Subcase 2.1), $p = 2$ and $|V(D_2)| \geq 3$ (Subcase 2.2) and $p = 2$ and $|V(D_2)| = 1$ (Subcase 2.3).

Subcase 2.1: Suppose that $p = 3$ and $|V(D_2)| = |V(D_3)| = 1$. Let $V(D_2) = \{v\}$ and $V(D_3) = \{w\}$. Observe that $D - v$ is strong, since $D_1 \to w$.

Hence, if $|N^+(s, D_1)| \geq 2$, two vertices of D_1 and v are non-separating vertices of D. So assume that $N^+(s, D_1) = \{u\}$.

If $v \to s$, the vertices v, w and at least one vertex of D_1 are non-separating vertices of D. Hence $s \to v$. Since $D_1 \to D_2$ and $s \to v$, it follows that $(D_1 - u) \to s$.

If $|V(D_1)| = 3$, it is easy to check that D is isomorphic to Q_6, a contradiction. Therefore $|V(D_1)| \geq 4$.

Note that D_1 is a strong subtournament of D on at least four vertices. By Corollary 1.2, the component D_1 has at least two non-separating vertices x and y. If $u \notin \{x, y\}$, the vertices x, y and v are non-separating vertices of D. So assume that D_1 has exactly two non-separating vertices and u is one of them.

Let $u_1 u_2 \ldots u_{n-3} u_1$ be a Hamiltonian cycle of D_1. By the induction hypothesis, we conclude that D_1 is isomorphic to Q_{n-3} and $u \in \{u_1, u_{n-3}\}$. If $u = u_{n-3}$, the vertices u_1, u_{n-4} and v are non-separating vertices of D. Hence $u = u_1$. But then D is isomorphic to Q_n as defined in Theorem 1.3 with $v = x_1$, $w = x_2$, $s = x_3$ and $x_{i+3} = u_i$ for $i = 1, 2, \ldots, n-3$, a contradiction.

Subcase 2.2: Suppose that $p = 2$ and $|V(D_2)| \geq 3$. If $|N^+(s, D_1)| \geq 2$ (or $|N^-(s, D_2)| \geq 2$), two vertices of D_1 (of D_2) and a vertex of D_2 (of D_1) are non-separating vertices of D. So assume that $N^+(s, D_1) = \{u\}$ and $N^-(s, D_2) = \{v\}$. Since $N^+(s, D_2) \neq \emptyset \neq N^-(s, D_1)$, it follows that $(D_1 - u) \to s \to (D_2 - v)$. Let $u_1 u_2 \ldots u_r u_1$ be a Hamiltonian cycle of D_1 and let $v_1 v_2 \ldots v_t v_1$ be a Hamiltonian cycle of D_2.

If $r = 3$, note that D_1 is isomorphic to Q_3. We may assume, without loss of generality, that $\{u_2, u_3\} \to s \to u_1$. Analogously, if $t = 3$, the component D_2 is isomorphic to Q_3 and we may assume, without loss of generality, that $v_3 \to s \to \{v_1, v_2\}$.

Assume that $r \geq 4$. Then the component D_1 has at least two non-separating vertices $\{x, y\}$ by Corollary 1.2. If $u \notin \{x, y\}$, the vertices x, y and a vertex of D_2 are non-separating vertices of D. Hence D_1 has exactly two non-separating vertices and u is one of them. By the induction hypothesis we conclude that D_1 is isomorphic to Q_r and $y \in \{u_1, u_r\}$. If $u = u_r$, the vertices u_1, u_{r-1} and a vertex of D_2 are non-separating vertices of D. Therefore $u = u_1$. Analogously, if $t \geq 4$, the component D_2 is isomorphic to Q_t with vertices $v_1, v_2, \ldots, v_t$ and $v = v_t$.

But then D is isomorphic to Q_n as defined in Theorem 1.3 with $x_i = v_i$ for $i = 1, 2, \ldots, t$, $x_{i+1} = s$ and $x_{i+2} = u_i$ for $i = 1, 2, \ldots, r$, a contradiction.

Subcase 2.3: Suppose that $p = 2$ and $|V(D_2)| = 1$. Then the single vertex v of D_2 is a non-separating vertex of D.

If $|N^+(s, D_1)| \geq 2$, two vertices of D_1 and v are non-separating vertices of D. So assume that $N^+(s, D_1) = \{u\}$. It follows that $(D_1 - u) \rightsquigarrow s$. Let $u_1 u_2 \ldots u_{n-2} u_1$ be a Hamiltonian cycle of D_1.

If $|V(D_1)| = 3$, note that D_1 is isomorphic to Q_3. Moreover, $n = 5$ and we may assume, without loss of generality, that $u_3 \to u_1$. If $\{u_2, u_3\} \to s \to u_1$, the local tournament D is isomorphic to Q_5 and if u_2 and s are not adjacent, the local tournament D is isomorphic to Q_5^*, both cases contradict our assumption.

So assume that $|V(D_1)| \geq 4$. In view of Corollary 1.2, the component D_1 has at least two non-separating vertices x, y. If $u \notin \{x, y\}$, the vertices x, y and v are non-separating vertices of D. Hence D_1 has exactly two non-separating vertices and u is one of them. By the induction hypothesis we conclude that D_1 is isomorphic to Q_{n-2} and $u \in \{u_1, u_{n-2}\}$. If $u = u_{n-2}$, the vertices u_1, u_{n-3} and v are non-separating vertices of D. Therefore $u = u_1$. It follows that $(D_1 - u) \to s$. But then D is isomorphic to Q_n as defined in Theorem 1.3 with $z = x_1$, $s = x_2$ and $x_{i+2} = y_i$ for $i = 1, 2, \ldots, n-2$, a contradiction. This completes the proof of this theorem. $\square$

In view of Theorem 1.5, Theorem 1.3 is an immediate corollary of the above theorem.

4 The number of cycles in local tournaments

In Section 3 we investigated the number of non-separating vertices of strongly connected local tournaments and characterized all strongly connected local tournaments on n vertices with at least $n+2$ arcs that have exactly two such vertices. In other words we characterized all strongly connected local tournaments on n

vertices with at least $n+2$ arcs that have exactly two cycles of length $n-1$. We reformulate Theorem 3.4 in these terms.

Theorem 4.1. *Let D be a strong local tournament on n vertices with at least $n+2$ arcs. Then D has exactly two cycles of length $n-1$ if and only if D is isomorphic to Q_n, Q_5^* or C_n^j for an integer $1 \leq j \leq n$.*

In this section we investigate Problem 1.7 and transfer Theorems 1.8 and 1.9 to the class of local tournaments.

4.1 Observations on the structure of local tournaments that are not round-decomposable

By refining the proof of Theorem 2.7 we can show that a strongly connected local tournament that is not a tournament has the following structure.

Theorem 4.2. *Let D be a strong local tournament which is not a tournament and not round-decomposable. Let S be a minimal separating set of D that satisfies the conditions of Theorem 2.7. Then one of the following possibilities holds.*

(a) There is a vertex $s \in S$ and vertices $x_1, x_2 \in V(D_2')$ with $x_1 \to s \to x_2$ and D has a Hamiltonian cycle C such that x_1 is the predecessor of x_2 on C;

(b) There is a vertex $x \in V(D_2')$ and vertices $s_1, s_2 \in S$ with $s_1 \to x \to s_2$ and D has a Hamiltonian cycle C such that s_1 is the predecessor of s_2 on C.

Proof. Let S be a minimal separating set of D such that $D-S$ is not a tournament, $D[S]$ is a tournament and the semicomplete decomposition of $D-S$ has exactly three components D_1', D_2' and D_3'. Let α, β, μ and ν be integers with $\lambda \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that

$$N^-(D_\alpha, D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha, D_\nu) \neq \emptyset$$
$$\text{or } N^-(D_\mu, D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu, D_\beta) \neq \emptyset,$$

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D-S$ and $D[S]$, respectively, and D_λ is the initial component of D_2'. By symmetry we may assume, without loss of generality, that $N^-(D_\alpha, D_\mu) \neq \emptyset$ and $N^+(D_\alpha, D_\nu) \neq \emptyset$. Let μ and ν be chosen such that $\nu - \mu$ is minimal. Furthermore, let $s_1 \in V(D_\mu)$, $s_2 \in V(D_\nu)$ and $\{x_1, x_2\} \subseteq V(D_\alpha)$ be vertices of D such that $s_1 \to x_1$ and $x_2 \to s_2$ (it is possible that $s_1 = s_2$ or $x_1 = x_2$).

If $s_1 \not\to D_\alpha$ ($D_\alpha \not\to s_2$), there exists a vertex $x \in V(D_\alpha)$ such that $x \to s_1 \to x^+$ ($x \to s_2 \to x^+$), i.e. (a) holds. So assume that $s_1 \to D_\alpha \to s_2$.

If $D_\alpha \not\to D_\nu$, there exists a vertex $x \in V(D_\alpha)$ and a vertex $s \in V(D_\nu)$ such that $s \to x \to s^+$, i.e. (b) holds. So assume that $D_\alpha \to D_\nu$.

Analogously, if $D_\mu \not\to D_\alpha$, there exists a vertex $x \in V(D_\alpha)$ and a vertex $s \in V(D_\mu)$ such that $x \to s \to x^+$, i.e. (a) holds. So assume that $D_\mu \to D_\alpha$.

If $\mu+1 = \nu$, note that $s_1 \to x_i \to s_2$ for $i = 1, 2$, i.e. (b) holds. So assume that $\mu+1 < \nu$. But then $N^-(D_{\mu+1}, D_\alpha) \neq \emptyset$ or $N^+(D_{\mu+1}, D_\alpha) \neq \emptyset$, both contradictions to the choice of μ and ν. □

Using the structure obtained in Theorem 4.2, we can show the following lemma which is a preparatory result for Theorem 4.5.

Lemma 4.3. *Let D be a strong local tournament on n vertices which is not a tournament. If D is not round-decomposable, then D has $n-5$ distinct vertices $v_1, v_2, \ldots, v_{n-5}$ such that for every $i = 1, 2, \ldots, n-5$ the digraph $D - v_i$ is a strong, not round-decomposable local tournament which is not a tournament.*

Proof. Let S be a minimal separating set of D that satisfies the conditions of Theorem 2.7. By Theorem 4.2, there exist integers α, β, μ and ν with $\lambda \le \alpha \le \beta \le p-1$ and $p+1 \le \mu \le \nu \le p+q$ such that either

(a) there is a vertex $s \in S$ and vertices $x_1, x_2 \in V(D'_2)$ with $x_1 \to s \to x_2$ and D has a Hamiltonian cycle C such that x_1 is the predecessor of x_2 on C or

(b) there is a vertex $x \in V(D'_2)$ and vertices $s_1, s_2 \in S$ with $s_1 \to x \to s_2$ and D has a Hamiltonian cycle C such that s_1 is the predecessor of s_2 on C,

where $D_1, D_2, \ldots, D_p$ and $D_{p+1}, D_{p+2}, \ldots, D_{p+q}$ are the strong decompositions of $D - S$ and $D[S]$, respectively, and D_λ is the initial component of D'_2. By symmetry we may assume, without loss of generality, that (a) holds.

In this case for every vertex $v \in (V(D'_2) \cup S) - \{s, x_1, x_2\}$ the digraph $D - v$ is a strong local tournament which is not a tournament and not round-decomposable. In addition, if $|V(D'_j)| \ge 3$ for an index $j \in \{1, 3\}$, then for every vertex $v \in V(D'_j)$, the digraph $D - v$ is a strong local tournament which is not a tournament and not round-decomposable. □

4.2 A generalization of Las Vergnas' Theorem

In this subsection we transfer Theorem 1.8 to the class of local tournaments. Firstly we consider strongly connected local tournaments that are not round-decomposable. The next example shows that Theorem 1.8 cannot be transferred directly to the class of local tournaments (cf. Figure 4).

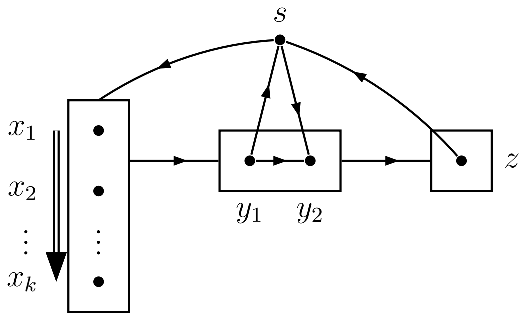

Figure 4: A class of local tournaments with $n-3$ cycles of length three.

Example 4.4. *Let $k \geq 1$ be an integer and let D be the local tournament with vertex set*

$$V = \{s\} \cup \{x_i \mid i = 1, 2, \ldots, k\} \cup \{y_1, y_2\} \cup \{z\}$$

and arc set

$$\begin{aligned} E = &\{x_i x_j \mid 1 \leq i < j \leq k\} \cup \{y_1 y_2\} \cup \{s x_i \mid i = 1, 2, \ldots, k\} \\ &\cup \{x_i y_j \mid i = 1, 2, \ldots, k,\ j = 1, 2\} \cup \{y_i z \mid i = 1, 2\} \\ &\cup \{zs\} \cup \{y_1 s\} \cup \{s y_2\}. \end{aligned}$$

Then $|V| = k + 4$ and the number of cycles of length 3 in D is $k + 1 = |V| - 3$.

However we can show the following result. Note that every strongly connected (local) tournament on less than five vertices is round-decomposable.

Theorem 4.5. *Let D be a strong local tournament on $n \geq 5$ vertices that is not round-decomposable. Then D has at least $n - r + 1$ cycles of length r for every $4 \leq r \leq n$.*

Proof. We shall prove the proposition by induction on n.

Induction basis. Let $n = 5$. Since D is strong, it has a Hamiltonian cycle by Theorem 2.1. Furthermore, since D is not round-decomposable, it has at least $n + 2$ arcs. Then D has at least two cycles of length four by Theorem 1.5.

Inductive step. Let $n \geq 6$. If D is not a tournament, it has a vertex x such that $D - x$ is a strong local tournament that is not round-decomposable by Lemma 4.3. If D is a tournament, it has a non-separating vertex x by Theorem 1.8. By the induction hypothesis, the digraph $D - x$ has at least $(n-1) - r + 1$ cycles of length r. Since D is either a tournament or a local tournament that is not round-decomposable and not a tournament, it is vertex pancyclic by Theorem 1.8 or 1.13. It follows that there is a cycle of length r through x in D. Therefore D has at least $n - r + 1$ cycles of length r. □

We now consider round-decomposable local tournaments.

Theorem 4.6. *Let D be a strongly connected and round-decomposable local tournament on $n \geq 5$ vertices with the round decomposition $R[D_1, D_2, \ldots, D_p]$, where $p \geq 3$. Let C_R be a shortest cycle of length $g(D)$ in R. Then*

(a) D has at least $n - r + 2$ cycles of length r for every $g(D) + 2 \leq r \leq n - 1$;

(b) D has at least $n - g(D)$ cycles of length $g(D) + 1 \leq n - 1$ with equality if and only if

(i) $|V(D_i)| = 1$ for every component $D_i \in V(C_R)$;

(ii) if $D_i \in V(C_R)$, then $D - x$ is not strong for every vertex $x \in V(D_i)$;

(iii) if $D_j \notin V(C_R)$, then $|V(D_j)| \leq g(D)$.

Proof. Let

$$C_R = D_1 D_{\alpha_2} D_{\alpha_3} \ldots D_{\alpha_g} D_1$$

be a cycle of length $g = g(D)$ in R, where $1 < \alpha_2 < \alpha_3 < \ldots < \alpha_g$. The cycle C_R induces a cycle

$$C = x_1 x_{\alpha_2} x_{\alpha_3} \ldots x_{\alpha_g} x_1$$

in D, where $x_i \in V(D_i)$. Note that every vertex $v \notin V(C)$ has a positive as well as a negative neighbor on C. Let S be an arbitrary set of vertices such that $v \notin V(C)$ for every vertex $v \in S$. Then all vertices of S can be inserted in C to construct a cycle of length $g(D) + |S|$ in D by Lemma 2.2. It follows that there are at least $\binom{n-g(D)}{s}$ cycles of length $g(D) + s$ in D, where $0 \leq s \leq n - g(D)$.

If $2 \leq s \leq n - g(D) - 1$, we derive $g(D) + 2 \leq g(D) + s = r \leq n - 1$ and

$$\binom{n - g(D)}{s} \geq n - g(D) \geq n - r + 2,$$

since $g(D) + 2 \leq r$. So (a) is true.

If $s = 1$, we deduce that D has at least

$$\binom{n - g(D)}{1} = n - g(D)$$

cycles of length $g(D) + s = g(D) + 1$. Note that all these cycles include at least one vertex of every component $D_i \in V(C_R)$. To prove (b) assume that one of the conditions (i), (ii) or (iii) is not satisfied.

Case 1. Suppose that $|V(D_i)| \geq 3$ for a component D_i of C_R. Then the cycle C_R induces at least $|V(D_i)|$ cycles of length $g(D)$ in D. Since $|V(D_i)|-1 \leq n-g(D)$ and $|V(D_i)| \geq 3$, it follows that there are at least

$$\begin{aligned}
&|V(D_i)|(n-g(D)+1-|V(D_i)|) + \binom{|V(D_i)|}{2} \\
&\qquad = n-g(D)+1+(|V(D_i)|-1)(n-g(D)+1) - \frac{|V(D_i)|^2+|V(D_i)|}{2} \\
&\qquad \geq n-g(D)+1+(|V(D_i)|-1)|V(D_i)| - \frac{|V(D_i)|^2+|V(D_i)|}{2} \\
&\qquad = n-g(D)+1+\frac{|V(D_i)|(|V(D_i)|-3)}{2} \\
&\qquad \geq n-g(D)+1
\end{aligned}$$

cycles of length $g(D)+1$ in D.

Case 2. Suppose that $D-x$ is strong for a vertex $x \in V(D_i)$, where D_i is a component of C_R. Then there exists a cycle of length $g(D)+1$ in D that includes no vertex of D_i. Therefore there are at least $n-g(D)+1$ cycles of length $g(D)+1$ in D.

Case 3. Suppose that $|V(D_j)| \geq g(D)+1$ for a component D_j that is not on C_R. Since D_j induces a strong subtournament of D, the component D_j is vertex pancyclic by Theorem 1.1. In particular there exists a cycle of length $g(D)+1$ in D_j and this cycle includes no vertex of D_i for every component D_i of C_R. Therefore there are at least $n-g(D)+1$ cycles of length $g(D)+1$ in D.

Assume now that the conditions (i)-(iii) are satisfied. In this case the cycle C_R induces a unique cycle C of length $g(D)$ in D. Due to the conditions (i)-(iii) there does not exist another cycle of length $g(D)$ in D. Therefore a cycle of length $g(D)+1$ in D consists of all vertices of C and another (arbitrary) vertex of D. It follows that D has exactly $n-g(D)$ cycles of length $g(D)+1$ and the proof of this theorem is complete. □

As an abbrevation we give the following definition.

Definition 4.7. *We define $\mathcal{R}^*$ to be the class of strongly connected and round-decomposable local tournaments that satisfy conditions (i)-(iii) of Theorem 4.6.(b).*

The following observations will be needed in the proof of Theorem 4.9.

Observation 4.8. *Let T be a strong tournament on $n \geq 5$ vertices. Then T has the following properties.*

(a) $T \notin \mathcal{R}^*$;

(b) If y is a vertex of T such that $T - y$ is strong, then $T - y \notin \mathcal{R}^*$.

Proof. Note that (b) is a direct consequence of (a). To prove the latter, assume that $T \in \mathcal{R}^*$ is a round-decomposable tournament with the round decomposition $R[D_1, D_2, \ldots, D_p]$. Then $g(T) = 3$ and, since every vertex of R is on a 3-cycle, we conclude that $|V(D_i)| = 1$ for every component D_i by Theorem 4.6.(b).(i). This means that $T = R$. But now Theorem 4.6.(b).(ii) implies that every vertex of T is a separating vertex. Using Corollary 1.2, we conclude that $n = 3$, a contradiction. □

In the next theorem we characterize all local tournaments for which the inequality in Theorem 4.5 is sharp. The proof follows the ideas of Las Vergnas' proof for tournaments [8].

Theorem 4.9. *Let D be a strong local tournament on $n \geq 5$ vertices with at least $n + 2$ arcs. If D is neither isomorphic to Q_n nor to Q_5^* nor to a member of $\mathcal{R}^*$, then D has at least $n - r + 2$ cycles of length r for every $g(D) + 1 \leq r \leq n - 1$.*

Proof. Note that D is vertex $g(D) + 1$-pancyclic by Theorem 1.13. Furthermore, if D is round-decomposable, it has at least $n - r + 2$ cycles of length r for every $g(D) + 1 \leq r \leq n - 1$ by Theorem 4.6. So we assume for the remaining part of the proof that D is not round-decomposable. We prove the proposition by induction on n.

Induction basis. Let $n = 5$. We have to show that D has at least three cycles of length four, in other words, that D has at least three non-separating vertices. This has been proved in Theorem 3.4.

Inductive step. Let $n \geq 6$. Note that D has at least $n - (n - 1) + 2 = 3$ cycles of length $n - 1$ by Theorem 3.4. Recall that if D is not a tournament, it has at least $n - 5 \geq 1$ non-separating vertices whose removal yields a strongly connected local tournament that is neither round-decomposable nor a tournament (cf. Lemma 4.3). If D is a tournament, it has at least three non-separating vertices by Theorem 3.4. Note that if y is one of these vertices, then $D - y \notin \mathcal{R}^*$ by Observation 4.8.

Now let x be a non-separating vertex that has the above properties.

Case 1: Suppose that $D - x$ is neither isomorphic to Q_{n-1} nor to Q_5^*. Then, by the induction hypothesis, $D - x$ has $(n - 1) - r + 2 = n - r + 1$ cycles of length r for $g(D - x) + 1 \leq r \leq n - 2$. Since $g(D - x) = g(D) = 3$ and D has a cycle of length r through x, the proof is complete in this case.

Case 2: Suppose that Case 1 does not hold, but $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and that D has two cycles of length r through x. Recall that Q_{n-1} and Q_5^* have exactly $(n - 1) - r + 1 = n - r$ cycles of length r for $4 \leq r \leq n - 2$. Then D has $n - r + 2$ cycles of length r and the proof of this case is complete.

Case 3: Suppose that neither Case 1 nor Case 2 holds, but $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and that D has exactly one cycle of length r through x. We consider two subcases depending on n.

Case 3.1: Suppose that $n \geq 8$. If D is not a tournament, it has three vertices u, v, w such that $D - u$, $D - v$ and $D - w$ are strongly connected local tournaments that are neither round-decomposable nor tournaments (cf. Lemma 4.3). If D is a tournament, it has three non-separating vertices u, v, w by Theorem 3.4.

We assume that $D - y \cong Q_{n-1}$ for every vertex $y \in \{u, v, w\}$ and that there is exactly one cycle of length r in D through each of the vertices u, v, w.

Let $\{y_1, y_2, \ldots, y_{n-1}\}$ be the vertex set of $D - u$ such that $y_i \to y_{i+1}$ and $y_k \to y_j$ and let $\{z_1, z_2, \ldots, z_{n-1}\}$ be the vertex set of $D - v$ such that $z_i \to z_{i+1}$ and $z_k \to z_j$ for $1 \leq i \leq n-2$ and $1 \leq j < k \leq n-1$ with $j \neq k+1$. Note that the only vertices that belong to exactly one cycle of length r in $D - u$ and $D - v$ are $\{y_1, y_{n-1}\}$ and $\{z_1, z_{n-1}\}$, respectively. It follows that $\{v, w\} = \{y_1, y_{n-1}\}$ and $\{u, w\} = \{z_1, z_{n-1}\}$. Let, without loss of generality, $y_1 = v$ and $y_{n-1} = w$. Since $d^-(w, D - u) = 1$, it follows that $d^-(w, D - v) \leq 2$. In addition, since $d^-(z_1, D - v) = n - 3 > 2$, it follows that $z_1 = u$ and $z_{n-1} = w$. Since $d^+(y_2, D - u) = 1$, it follows that $d^-(y_2, D - v) \leq 2$. The latter implies that $y_2 \in \{z_2, z_3\}$.

Case 3.1.1: Suppose that $y_2 = z_3$. Note that v is dominated by $V(D) - \{u, y_2\}$ and dominates y_2. It follows that $vz_3z_4 \ldots z_rz_2v$ and $vz_3z_4 \ldots z_{r-1}z_1z_2v$ are two cycles of length r through v, a contradiction.

Case 3.1.2: Suppose that $y_2 = z_2$. Note again that v is dominated by $V(D) - \{u, y_2\}$ and dominates y_2. Since D is a local tournament, it follows that $z_1 = u$ and v are adjacent. If v is dominated by u, the cycles $vz_2z_3 \ldots z_{r-1}z_1v$ and $vz_2z_3 \ldots z_rv$ are two cycles of length r through v, a contradiction. So assume that v dominates u. But then $vz_1z_2 \ldots z_{r-1}v$ and $vz_2z_3 \ldots z_rv$ are two cycles of length r through v, again a contradiction.

Case 3.2: Suppose that $n \in \{6, 7\}$. We assume that $D - x \cong Q_{n-1}$ or $D - x \cong Q_5^*$ and that there is exactly one cycle of length r in D through x. Let $\{y_1, y_2, \ldots, y_{n-1}\}$ be the vertex set of $D - x$ such that $y_i \to y_{i+1}$ and $y_k \to y_j$ for $1 \leq i \leq n-2$ and $1 \leq j < k \leq n-1$ with $j \neq k+1$ (if $D - x \cong Q_5^*$, there is no arc between y_2 and y_4). Since D is strong, the vertex x has at least one out- and one in-neighbor in $D - x$. It follows that there exists an index $1 \leq i \leq n$ such that $y_i \to x \to y_{i+1}$. By Theorem 3.4 we may assume that $r \leq n - 2$. Hence we consider three remaining cases.

Case 3.2.1: Suppose that $r = 4$ and $n = 6$. We show that there exist two cycles of length four through x without using the arc y_4y_2. Note that the case $i = 1$ and $i = 4$ as well as the cases $i = 2$ and $i = 3$ are symmetrical.

If $i = 1$, the cycle $xy_2y_3y_1x$ is a 4-cycle through x. In addition, since $\{x, y_5\} \to$

y_2, the vertices x and y_5 are adjacent. If $x \rightarrow y_5$, the cycle $xy_5y_3y_1x$ is a second 4-cycle through x, a contradiction. So assume that $y_5 \rightarrow x$. Then, since $y_5 \rightarrow \{x, y_3\}$, the vertices x and y_3 are adjacent. If $x \rightarrow y_3$, the cycle $xy_3y_4y_5x$ is a second 4-cycle through x, a contradiction. So assume that $y_3 \rightarrow x$. Since $y_3 \rightarrow \{x, y_4\}$, the vertices x and y_4 are adjacent. If $y_4 \rightarrow x$, the cycle $xy_2y_3y_4x$ is a second 4-cycle through x and if $x \rightarrow y_4$, the cycle $xy_4y_5y_1x$ is a second 4-cycle through x, both possibilities are contradictions.

If $i = 2$, the cycle $xy_3y_1y_2x$ is a 4-cycle through x. In addition, since $\{x, y_5\} \rightarrow y_3$, the vertices x and y_5 are adjacent. If $x \rightarrow y_5$, the cycle $xy_5y_1y_2x$ is a second 4-cycle through x and if $y_5 \rightarrow x$, the cycle $xy_3y_4y_5x$ is a second 4-cycle through x, both possibilities are contradictions.

If $i = 5$, there is an arc between x and each of the vertices y_j for $j = 2, 3, 4$, since $\{x, y_3, y_4\} \rightarrow y_1$ and $y_5 \rightarrow \{x, y_2\}$. If $x \rightarrow \{y_2, y_3, y_4\}$ or if $\{y_2, y_3, y_4\} \rightarrow x$, the local tournament D is isomorphic to Q_6, a contradiction. If $y_4 \rightarrow x \rightarrow \{y_2, y_3\}$ or if $\{y_3, y_4\} \rightarrow x \rightarrow y_2$, the cycle $xy_2y_3y_4x$ is a 4-cycle through x. A second cycle of length four is given by $xy_3y_4y_5x$ and $xy_1y_2y_3x$, respectively, again a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.1 is complete.

Case 3.2.2: Suppose that $r = 4$ and $n = 7$. Note that the cases $i = 1$ and $i = 5$ are symmetrical.

If $2 \leq i \leq 4$, the cycles $xy_{i+1}y_{i-1}y_ix$ and $xy_{i+1}y_{i+2}y_ix$ are two cycles of length four through x, a contradiction.

If $i = 1$, the cycle $xy_2y_3y_1x$ is a 4-cycle through x. In addition, since $\{x, y_4\} \rightarrow y_2$, the vertices x and y_4 are adjacent. If $x \rightarrow y_4$, the cycle $xy_4y_5y_1x$ is a second 4-cycle through x and if $y_4 \rightarrow x$, the cycle $xy_2y_3y_4x$ is a second 4-cycle through x, both possibilities are contradictions.

If $i = 6$, there is an arc between x and each of the vertices y_j for $j = 2, 3, 4, 5$, since $\{x, y_4, y_5\} \rightarrow y_1$ and $y_6 \rightarrow \{x, y_2, y_3\}$. If $x \rightarrow \{y_2, y_3, y_4, y_5\}$ or if $\{y_2, y_3, y_4, y_5\} \rightarrow x$, the digraph D is isomorphic to Q_7, a contradiction. If $y_5 \rightarrow x \rightarrow \{y_2, y_3, y_4\}$, the cycles $xy_3y_4y_5x$ and $xy_4y_5y_6x$ are two cycles of length four through x; if $\{y_3, y_4, y_5\} \rightarrow x \rightarrow y_2$, the cycles $xy_2y_3y_4x$ and $xy_1y_2y_3x$ are two cycles of length four through x; and if $\{y_4, y_5\} \rightarrow x \rightarrow \{y_2, y_3\}$, the cycles $xy_2y_3y_4x$ and $xy_3y_4y_5x$ are two cycles of length four through x. Thus, all three possibilities yield a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.2 is complete.

Case 3.2.3: Suppose that $r = 5$ and $n = 7$. Note that the cases $i = 1$ and $i = 5$ are symmetrical.

If $i = 1$, the cycle $xy_2y_3y_4y_1x$ is a 5-cycle through x. In addition, since $\{x, y_5\} \rightarrow y_2$, the vertices x and y_5 are adjacent. If $y_5 \rightarrow x$, the cycle $xy_2y_3y_4y_5x$ is a second 5-cycle through x and if $x \rightarrow y_5$, the cycle $xy_5y_3y_4y_1x$ is a second 5-cycle through x. Both possibilities yield a contradiction.

If $i = 2$, the cycles $xy_3y_4y_1y_2x$ and $xy_3y_4y_5y_2x$ are two cycles of length five through x, a contradiction.

If $i \in \{3, 4\}$, the cycles $xy_{i+1}y_{i-2}y_{i-1}y_ix$ and $xy_{i+1}y_{i+2}y_{i-1}y_ix$ are two cycles of length five through x, a contradiction.

If $i = 6$, there is an arc between x and each of the vertices y_j for $j = 2, 3, 4, 5$, since $\{x, y_4, y_5\} \to y_1$ and $y_6 \to \{x, y_2, y_3\}$. If $x \to \{y_2, y_3, y_4, y_5\}$ or if $\{y_2, y_3, y_4, y_5\} \to x$, the local tournament D is isomorphic to Q_7, a contradiction. If $y_5 \to x \to \{y_2, y_3, y_4\}$ or if $\{y_4, y_5\} \to x \to \{y_2, y_3\}$, the cycles $xy_2y_3y_4y_5x$ and $xy_3y_4y_5y_6x$ are two cycles of length five through x, a contradiction. If $\{y_3, y_4, y_5\} \to x \to y_2$, the cycles $xy_2y_3y_4y_5x$ and $xy_1y_2y_3y_4x$ are two cycles of length five through x, again a contradiction. Since we have now considered all possible cases, the proof of Case 3.2.3 and therefore the proof of this theorem is complete. □

We would like to add the following remark.

Remark 4.10. *Let D be a strong local tournament that is not round-decomposable and not a tournament. If D does not fulfill the conclusion of Theorem 4.9, then D is isomorphic to Q_5^*.*

As a corollary of Theorems 4.6 and 4.9 we derive Theorem 1.9.

Corollary 4.11 (Las Vergnas [8] 1975)**.** *Let T be a strong tournament on $n \geq 5$ vertices. Then T has at least $n - r + 2$ cycles of length r for every $4 \leq r \leq n - 1$ unless T is isomorphic to Q_n.*

References

[1] J. Bang-Jensen, *Locally semicomplete digraphs: a generalization of tournaments*, J. Graph Theory **14** (1990), no. 3, 371–390.

[2] J. Bang-Jensen, Y. Guo, G. Gutin, and L. Volkmann, *A classification of locally semicomplete digraphs*, Discrete Math. **167/168** (1997), 101–114, 15th British Combinatorial Conference (Stirling, 1995).

[3] M. Burzio and D. C. Demaria, *Hamiltonian tournaments with the least number of 3-cycles*, J. Graph Theory **14** (1990), no. 6, 663–672.

[4] R. J. Douglas, *Tournaments that admit exactly one Hamiltonian circuit*, Proc. London Math. Soc. (3) **21** (1970), 716–730.

[5] Y. Guo and L. Volkmann, *Connectivity properties of locally semicomplete digraphs*, J. Graph Theory **18** (1994), no. 3, 269–280.

[6] G. Korvin, *Some combinatorial problems on complete directed graphs*, Theory of Graphs (Internat. Sympos., Rome, 1966), Gordon and Breach, New York, 1967, pp. 197–213.

[7] K. Kotani, *Nonseparating vertices in tournaments with large minimum degree*, Australas. J. Combin. **36** (2006), 27–38.

[8] M. Las Vergnas, *Sur le nombre de circuits dans un tournoi fortement connexe*, Cahiers Centre Études Recherche Opér. **17** (1975), no. 2-3-4, 261–265, Colloque sur la Théorie des Graphes (Paris, 1974).

[9] D. Meierling and L. Volkmann, *On the number on nonseparating vertices in strongly connected local tournaments*, Australas. J. Combin. **39** (2007), 301–315.

[10] J. W. Moon, *On subtournaments of a tournament*, Canad. Math. Bull. **9** (1966), 297–301.