On the number of nonseparating vertices in strongly connected local tournaments

Dirk Meierling and Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
e-mail: {meierling,volkm}@math2.rwth-aachen.de

Abstract

A local tournament is an oriented graph such that the positive as well as the negative neighborhood of every vertex induces a tournament. Let $p \geq 2$ be an integer and let T be a strongly connected tournament such that every vertex has at least p positive neighbors and at least p negative neighbors. Kotani [7] showed that T has at least k vertices $x_1, x_2, \ldots, x_k$, where $k = \min\{|V(D)|, 4p-2\}$, such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected. In this paper we shall show that this proposition is true for local tournaments, thereby generalizing Kotani's result.

1 Terminology and introduction

All digraphs mentioned here are finite without loops and multiple arcs. For a digraph D, we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The number $|V(D)|$ is the *order* of the digraph D. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$. By $D - A$ we denote the digraph $D[V(D) - A]$. If $A = \{x\}$ is a single vertex, then we write $D - x$ instead of $D - \{x\}$.

If $xy \in E(D)$, then y is a *positive neighbor* of x and x is a *negative neighbor* of y, and we also say that x *dominates* y, denoted by $x \to y$. If A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B in D, then we say that A *dominates* B, denoted by $A \to B$. The *outset* (*inset*) $N^+(x)$ ($N^-(x)$) of a vertex x is the set of positive (negative) neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A, and the inset $N^-(A, B)$ is defined analogously. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. The *minimum outdegree* $\delta^+(D)$ and the *minimum indegree* $\delta^-(D)$ of D are given by $\min\{d^+(x) \mid x \in V(D)\}$ and $\min\{d^-(x) \mid x \in V(D)\}$, respectively. Analogously, we define the *maximum outdegree* $\Delta^+(D) = \max\{d^+(x) \mid x \in V(D)\}$ and the *maximum indegree* $\Delta^-(D) = \max\{d^-(x) \mid x \in V(D)\}$ of D. In addition, let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the *minimum degree* and $\Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ be the *maximum degree* of D.

Throughout this paper, directed cycles and paths are simply called *cycles* and *paths*. A cycle (path) of order k is a *k-cycle* or a *k-path*, respectively. If C is a cycle of a digraph D with order $|V(D)|$, then C is called a *Hamiltonian cycle*. Let $P = x_1x_2 \ldots x_k$ be a path or $C = x_1x_2 \ldots x_kx_1$ be a cycle of D with order k. Then $P[x_i, x_j]$ (or $C[x_i, x_j]$, respectively), where $1 \leq i, j \leq k$, denotes the subpath $x_ix_{i+1} \ldots x_j$ of P (or C, respectively) with *initial vertex* x_i and *terminal vertex* x_j.

We speak of a *connected digraph* if the underlying graph is connected. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of D is a maximal induced strong subdigraph of D.

A vertex x is called a *nonseparating vertex* of a strong digraph D if $D - x$ is strong.

A digraph is *semicomplete* if for any two different vertices x and y, there is at least one arc between them. A *tournament* is a semicomplete digraph without 2-cycles. A digraph D is *locally semicomplete* if $D[N^+(x)]$ as well as $D[N^-(x)]$ are semicomplete for every vertex x of D. A *local tournament* is a digraph without 2-cycles such that the inset as well as the outset of every vertex induces a tournament.

We speak of an *r-regular tournament* T if $\delta(T) = \Delta(T) = r$. Similarly, an *almost-regular tournament* is a tournament such that $|\Delta(T) - \delta(T)| \leq 1$.

Throughout this paper all subscripts are taken modulo the corresponding number.

Local tournaments were introduced by Bang-Jensen [1] in 1990. In transferring the general adjacency only to vertices that have a common positive or negative neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning this class of digraphs, in particular the Ph.D. theses of Guo [3] and Huang [5] handeled this subject in detail. For more information concerning different generalizations of tournaments, the reader may be refered to the survey article of Bang-Jensen and Gutin [2].

In 1966, Moon [9] proved the following well-known result.

Theorem 1.1 (Moon [9] 1966). *If T is a strong tournament, then every vertex of T belongs to a cycle of length k for $k = 3, 4, \ldots, |V(T)|$.*

The next well-known proposition follows directly from Theorem 1.1. It was formulated and proved by Korvin [6] in 1967.

Corollary 1.2 (Korvin [6] 1967). *If T is a strong tournament with $|V(T)| \geq 4$, then T contains at least two nonseparating vertices.*

In 1975, Las Vergnas [8] determined all strong tournaments with three nonseparating vertices.

Theorem 1.3 (Las Vergnas [8] 1975). *A strong tournament T on n vertices has at least three nonseparating vertices, unless T is isomorphic to Q_n, where Q_n is the tournament of order n consisting of a path $v_1 v_2 \ldots v_n$ and all arcs $v_i v_j$ for $i > j + 1$.*

Bang-Jensen [1] and Guo, Volkmann [4] proved the following related results for locally semicomplete digraphs.

Theorem 1.4 (Bang-Jensen [1] 1990). *Let D be a strong locally semicomplete digraph that is not a cycle. Then D has a nonseparating vertex.*

Theorem 1.5 (Guo and Volkmann [4] 1994). *Let D be a strong locally semicomplete digraph on $n \geq 4$ vertices. If D has at least $n + 2$ arcs, then it contains at least two nonseparating vertices.*

In 2006, Kotani [7] gave the following variants of Corollary 1.2 for tournaments with minimum degree at least two.

Theorem 1.6 (Kotani [7] 2006). *Let T be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$, then T has at least $k = \min\{|V(T)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 1.7 (Kotani [7] 2006). *Let T be a strong tournament and let $p \geq 2$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p$, then T has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 1.8 (Kotani [7] 2006). *Let T be a strong tournament and let $p \geq 3$ be an integer. If $\delta(T) \geq p$ and $|V(T)| \geq 4p+1$, then T has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ is strong for $i = 1, 2, \ldots, k$.*

In this paper, we will show that Theorem 1.6 is true for local tournaments and characterize the class of local tournaments that do not fulfill Theorems 1.7 and 1.8, thereby generalizing Kotani's results.

2 Preliminaries

The following known results play an important role in our proofs.

Theorem 2.1 (Bang-Jensen [1] 1990). *If D is a strong local tournament with $|V(D)| \geq 3$, then D has a Hamiltonian cycle.*

Lemma 2.2 (Bang-Jensen [1] 1990). *Let D be a local tournament containing a cycle $C = u_1 u_2 \ldots u_k u_1$. If there exists a vertex $v \in V(D) - V(C)$ such that $N^+(v, C) \neq \emptyset$ (or $N^-(v, C) \neq \emptyset$), then either $v \rightarrow C$ ($C \rightarrow v$, respectively) or $u_i \rightarrow v \rightarrow u_{i+1}$ for some integer $1 \leq i \leq k$, i.e., there exists a cycle C' in D such that $V(C') = V(C) \cup \{v\}$.*

Theorem 2.3 (Bang-Jensen [1] 1990). *Let D be a connected local tournament that is not strong. Then the strong components of D can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from D_j to D_i for $j > i$ and D_i dominates D_{i+1} for $i = 1, 2, \ldots, p-1$.*

We call the ordering of the strong components as in Theorem 2.3 the *strong decomposition* of D. The subdigraphs D_1 and D_p are the *initial* and *terminal component* of D, respectively.

The next three lemmas will be frequently used in Section 3.

Lemma 2.4. *Let D be a strong digraph of order n and let q be an integer such that $1 \leq q \leq n-1$. Suppose that for every subset $X \subseteq V(D)$ with $1 \leq |X| \leq q$ there exists a cycle C of order $n-1$ in D such that $X \subseteq V(C)$. Then D contains at least $q+1$ vertices $x_1, x_2, \ldots, x_{q+1}$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, q+1$.*

Proof. Let $X = \{x \in V(D) \mid D - x \text{ is strong}\}$ be the set of nonseparating vertices of D. Suppose to the contrary that $|X| \leq q$. By the assumption of this lemma, there exists an $(n-1)$-cycle C in D such that $X \subseteq V(C)$. Let $\{v\} = V(D) - V(C)$. On the one hand this implies that $v \notin X$, but on the other hand C is a Hamiltonian cycle of $D - v$ and thus, $D - v$ is strong. It follows that $v \in X$, a contradiction. □

Lemma 2.5. *Let D be a digraph of order n without cycles of length two and let $r \geq 1$ be an integer. If $\delta^+(D) \geq r$ or $\delta^-(D) \geq r$, then $|V(D)| \geq 2r+1$.*

Proof. Suppose, without loss of generality, that $\delta^+(D) \geq r$. It follows that

$$\frac{n(n-1)}{2} \geq \sum_{x \in V(D)} d^+(x) \geq nr$$

which finally implies that $n \geq 2r+1$. $\square$

Lemma 2.6. *Let D be a digraph of order n without cycles of length two and let $r \geq 1$ be an integer. If $\delta^+(D) \geq r$ and there exists a vertex $x \in V(D)$ such that $d^+(x) \geq r+1$ or if $\delta^-(D) \geq r$ and there exists a vertex $x \in V(D)$ such that $d^-(x) \geq r+1$, then $|V(D)| \geq 2r+2$.*

Proof. Suppose, without loss of generality, that $\delta^+(D) \geq r$ and that there exists a vertex $x \in V(D)$ such that $d^+(x) \geq r+1$. It follows that

$$\frac{n(n-1)}{2} \geq \sum_{x \in V(D)} d^+(x) \geq r+1+(n-1)r = nr+1.$$

Since n is an integer, it follows that $n \geq 2r+2$. $\square$

3 Main results

We begin this section by showing that Theorems 1.7 and 1.8 do not hold for local tournaments in general.

Definition 3.1. *Let $p \geq 2$ be an integer. Let T_1 and T_2 be two $(p-1)$-regular tournaments. Let u, v be two vertices such that $u, v \notin V(T_i)$ for $i = 1, 2$.*

We define $\mathcal{T}^*$ as the set of all local tournaments D with vertex set $V(D) = V(T_1) \cup V(T_2) \cup \{u, v\}$ The arc set $E(D)$ is defined as follows.

- *D contains all arcs of T_1 and T_2,*
- *$u \to T_1 \to v \to T_2 \to u$ and*
- *D contains the arc uv or the arc vu or there is no arc between u and v in D.*

Let $p \geq 3$ be an integer. Let T_1 be defined as above and let T_0 be an almost regular tournament on $2p$ vertices such that $p-1 \leq \delta(T_0), \Delta(T_0) \leq p$. Let u, v be two vertices such that $u, v \notin V(T_i)$ for $i = 0, 1$. We define $\mathcal{T}^{**}$ as the set of all local tournaments D with vertex set $V(D) = V(T_0) \cup V(T_1) \cup \{u, v\}$. The arc set $E(D)$ is defined as follows.

- *D contains all arcs of T_0 and T_1,*
- *$u \to T_1 \to v \to T_0 \to u$ and*
- *D contains the arc uv or the arc vu or there is no arc between u and v in D.*

Remark 3.2. *a) Let $D \in \mathcal{T}^*$ be a local tournament. Then $|V(D)| = 4p$, $\delta(D) = p$ and $V(T_1) \cup V(T_2)$ is the set of nonseparating vertices of D, i.e., D does not fulfill Theorem 1.7.*

b) Let $D \in \mathcal{T}^{**}$ be a local tournament. Then $|V(D)| = 4p + 1$, $\delta(D) = p$ and $V(T_0) \cup V(T_1)$ is the set of nonseparating vertices of D, i.e., D does not fulfill Theorem 1.8.

In this section we characterize $\mathcal{T}^*$ ($\mathcal{T}^{**}$) as the class of local tournaments that are exceptions for Theorem 1.7 (Theorem 1.8, respectively).

Throughout this section, let $p \geq 2$ be an integer and let D be a strong local tournament such that $\delta(D) \geq p$. Let X be a subset of $V(D)$ such that $\emptyset \neq X \neq V(D)$. Obviously D contains a strong induced subdigraph D' such that $V(D) - V(D') - X \neq \emptyset$ (any vertex $x \in X$ satisfies this condition). Assume now that we have chosen a strong induced subdigraph D' of D under the following conditions:

A. $V(D) - V(D') - X \neq \emptyset$,

B. under condition A: $|V(D') \cap X|$ is maximal and

C. under condition B: $|V(D')|$ is maximal.

Note that, since D' is a strong local tournament, according to Theorem 2.1, the digraph D' has a Hamiltonian cycle $C = v_1 v_2 \dots v_k v_1$ (if $k = 1$, then, for the sake of simplicity, the single vertex v_1 will in the following be called a Hamiltonian cycle). Let

$$Y = V(D) - V(C) - X$$

and define

$$\begin{aligned} X^+ &= \{v \in X - V(C) \mid C \rightarrow v\}, \\ X^- &= \{v \in X - V(C) \mid v \rightarrow C\}, \\ \hat{X} &= X - V(C) - X^+ - X^-, \\ Y^+ &= \{v \in Y - V(C) \mid C \rightarrow v\}, \\ Y^- &= \{v \in Y - V(C) \mid v \rightarrow C\} \text{ and} \\ \hat{Y} &= Y - Y^+ - Y^-. \end{aligned}$$

Note that, according to these definitions,

$$N^+(X^+, C) = N^-(X^-, C) = N^+(Y^+, C) = N^-(Y^-, C) = \emptyset \quad (1)$$

and $D[V(C)]$ is a tournament if $X^+ \cup X^- \cup Y^+ \cup Y^- \neq \emptyset$.

Using this notation, we prove the following claims.

Claim 1. $\hat{X} = \{x \in X - V(C) \mid N^+(x, C) = N^-(x, C) = \emptyset\}$.

Proof. Assume that there exists a vertex $x \in \hat{X}$ such that $N^+(x, C) \neq \emptyset$. Then, according to Lemma 2.2, either $x \rightarrow C$ or there exists a cycle C' in D such that $V(C') = V(C) \cup \{x\}$. But the first possibility is a contradiction to the definition of $\hat{X}$ and the latter contradicts the choice of D'. It follows that $N^+(x, C) = \emptyset$. We can analogously show that $N^-(x, C) = \emptyset$. □

Claim 2. *Let $y \in \hat{Y}$ be an arbitrary vertex.*

a) If $N^+(y, C) \neq \emptyset$*, then* $N^+(y, \hat{X}) = \emptyset$*.*

b) If $N^-(y, C) \neq \emptyset$*, then* $N^-(y, \hat{X}) = \emptyset$*.*

Proof. Assume to the contrary that there exists a vertex $y \in \hat{Y}$ that has positive (or negative) neighbors both in C and $\hat{X}$. Using the local tournament property of D, it follows that these neighbors are adjacent, a contradiction to Claim 1. □

Analogously to Claim 1, we can prove the following claim.

Claim 3. *If* $|Y| \geq 2$*, then* $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$*.*

Claim 4. $N^+(\hat{X}, X^+) = N^+(\hat{X}, Y^+) = N^-(\hat{X}, X^-) = N^-(\hat{X}, Y^-) = \emptyset$*.*

Proof. Assume that D has an arc x_1x_2 such that $x_1 \in \hat{X}$ and $x_2 \in X^+$. Then x_2 has negative neighbors both in $V(C)$ and $\hat{X}$. Since D is a local tournament, there exists an arc between C and x_1, a contradiction to Claim 1. It follows that $N^+(\hat{X}, X^+) = \emptyset$. We can analogously solve the other three cases. □

Using Claim 3, we can prove the following claim analogously to Claim 4.

Claim 5. *If* $|Y| \geq 2$*, then* $N^+(\hat{Y}, X^+) = N^+(\hat{Y}, Y^+) = N^-(\hat{Y}, X^-) = N^-(\hat{Y}, Y^-) = \emptyset$*.*

Claim 6. $|Y^+|, |Y^-| \leq 1$*.*

Proof. Assume to the contrary that $|Y^+| \geq 2$. Let $P = z_0z_1 \ldots z_r$ be a shortest path in D such that $z_0 \in Y^+$ and $N^+(z_r, C) \neq \emptyset$. Then $|V(P) \cap Y^+| = 1$. We may assume, without loss of generality, that $z_r \to v_1$. But then $C' = C[v_1, v_k]Pv_1$ is a cycle with $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$ which contradicts the choice of D'. It follows that $|Y^+| \leq 1$. We can analogously show that $|Y^-| \leq 1$. □

Lemma 3.3. *Let* D *be a strong local tournament and let* $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ *be defined as in the paragraph proceeding Claim 1. Let* $u, w \in V(D) - V(C)$ *be two vertices of* D *such that* $N^+(w, C) \neq \emptyset$ *and* $N^-(u, C) \neq \emptyset$ *and let* P *be a path in* $V(D) - V(C)$ *with initial vertex* u *and final vertex* w*. Then* $Y \subseteq V(P)$ *and*

(i.) if $w \to C \to u$*, then* $V(C) \subseteq X$*,*

(ii.) if $N^-(w, C) \neq \emptyset$*, then* $\hat{Y} = \{w\}$*,*

(iii.) if $N^+(u, C) \neq \emptyset$*, then* $\hat{Y} = \{u\}$*.*

Proof. By Lemma 2.2, it follows that either $w \to C$ or $N^-(w, C) \neq \emptyset$ and that either $C \to u$ or $N^+(u, C) \neq \emptyset$.

Case 1: Suppose that $w \to C \to u$. If $Y - V(P) \neq \emptyset$, then the cycle $C' = v_kPC[v_1, v_k]$ yields a contradiction to the choice of D', where D' is defined as in the paragraph preceeding Claim 1, since $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$. Therefore $Y \subseteq V(P)$. Now assume to the contrary that $V(C) - X \neq \emptyset$, say $v_k \notin X$. Then $C' = v_{k-1}PC[v_1, v_{k-1}]$ is a cycle with

$|V(C')| > |V(C)|$ and $V(C) \cap X \subseteq V(C') \cap X$ which contradicts the choice of D'. It follows that $V(C) \subseteq X$.

Case 2: Suppose that $N^-(w, C) \neq \emptyset$. Then, according to (1), $w \notin X^+ \cup Y^+ \cup X^- \cup Y^-$. Furthermore, in view of Claim 1, $w \notin \hat{X}$ which implies that $w \in \hat{Y}$. Using Claim 3, we conclude that $Y = \hat{Y} = \{w\}$ and, in particular, $Y \subseteq V(P)$. We can analogously solve the case $N^+(u, C) \neq \emptyset$. □

We get the following two claims as immediate corollaries of Lemma 3.3.

Claim 7. $N^+(X^+, X^-) = \emptyset$.

Claim 8. *If* $|Y^+| = |Y^-| = 1$*, then* $N^+(X^+, Y^-) = N^+(Y^+, X^-) = \emptyset$.

Until the end of Lemma 3.6, we assume now that $X - V(C) \neq \emptyset$.

Lemma 3.4. *Let* D *be a strong local tournament and let* $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ *be defined as above. Let* $X - V(C) \neq \emptyset$*. If* $|Y| \geq 2$*, then* $X^+ \cup Y^+ \neq \emptyset$ *and* $X^- \cup Y^- \neq \emptyset$*.*

Proof. Assume to the contrary that $X^- \cup Y^- = \emptyset$. Note that, according to Claim 6, $|Y^+|, |Y^-| \leq 1$. Since $|Y| \geq 2$ and $Y^- = \emptyset$, we conclude that $\hat{Y} \neq \emptyset$. In view of Claim 1, $N^+(\hat{X}, C) = \emptyset$ which implies $N^+(\hat{Y}, C) \neq \emptyset$, since D is strong. This is a contradiction to Claim 3. We can analogously show that $X^+ \cup Y^+ \neq \emptyset$. □

Lemma 3.5. *Let* D *be a strong local tournament and let* $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ *be defined as above. Let* $\delta(D) \geq p \geq 2$ *and let* $X - V(C) \neq \emptyset$*. If* $|X| = |V(D)| - 1$*, then* $|X| \geq 4p - 2$*.*

Proof. Note that $V(C) \subseteq X$. Let $\{y\} = V(D) - X = Y$.

Case 1. Suppose that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$. Note that $N^+(X^+, X^-) = N^+(X^+, C) = N^-(X^-, X^+) = N^-(X^-, C) = \emptyset$. Using Lemma 2.5, it follows that $|X^+| \geq 2p - 1$ and $|X^-| \geq 2p - 1$ and thus,

$$|X| \geq |V(C)| + |X^+| + |X^-| \geq 4p - 1.$$

Case 2. Suppose that $X^+ = \emptyset$. Since $|Y| = |V(D) - V(C) - X| = 1$, we consider the three subcases $|\hat{Y}| = 1$, $|Y^+| = 1$ and $|Y^-| = 1$. We shall show first that $|V(C)| \geq 2p - 1$.

If $Y = \hat{Y} = \{y\}$, then $N^+(C) - V(C) \subseteq \{y\}$ according to Claim 1 and (1). Using Lemma 2.5, we conclude that $|V(C)| \geq 2p - 1$.

If $Y = Y^+ = \{y\}$, then note that $N^+(\hat{X}, C) = \emptyset$ according to Claim 1. Since D is strong, it follows that $X^- \neq \emptyset$. In view of Claim 1, $N^+(C, \hat{X}) = \emptyset$ and thus, $N^+(C) - V(C) \subseteq \{y\}$. Using Lemma 2.5, it follows that $|V(C)| \geq 2p - 1$. In the case that $Y = Y^- = \{y\}$ we can show analogously that $|V(C)| \geq 2p - 1$.

Note that $N^-(X^-, C) = \emptyset$ by definition and $N^-(\hat{X}, C) = \emptyset$ by Claim 1. Using Lemma 2.5, it follows that $|X^- \cup \hat{X}| \geq 2p - 1$ and thus,

$$|X| \geq |V(C)| + |X^+ \cup \hat{X}| \geq 4p - 2.$$

The case $X^- = \emptyset$ can be solved analogously which completes the proof of this lemma. □

Lemma 3.6. *Let D be a strong local tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as above. Let $X - V(C) \neq \emptyset$ and let $|X| \leq |V(D)| - 2$.*

a) If $\delta(D) \geq p \geq 2$, then $|X| \geq 4p - 1$ or $|X| = 4p - 2$ and $D \in \mathcal{T}^*$.

b) If $\delta(D) \geq p \geq 3$, then $|X| \geq 4p$ or $|X| = 4p - 1$ and $D \in \mathcal{T}^{**}$ or $|X| = 4p - 2$ and $D \in \mathcal{T}^*$.

Proof. Recall that $|Y| \geq 1$. If $|Y| = 1$, we conclude $|V(C) - X| \geq 1$, since $|X| \leq |V(D)| - 2$. If $|Y| \geq 2$, we derive by Lemma 3.4 that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$. Since D is strong there exists an $(X^+ \cup Y^+)$-$(X^- \cup Y^-)$-path in D that omits C. Together with Lemma 3.3, this implies that $V(C) \subseteq X$. We will now consider the cases $|Y| = 1$ and $|V(C) - X| \geq 1$ (Case 1), $|Y| \geq 2$, $V(C) \subseteq X$ and $\hat{Y} = \emptyset$ (Case 2) and $|Y| \geq 2$, $V(C) \subseteq X$ and $\hat{Y} \neq \emptyset$ (Case 3).

Case 1. Suppose that $|Y| = 1$ and $|V(C) - X| \geq 1$.

Case 1.1. Suppose that $X^+ \cup Y^+ \neq \emptyset$ and $X^- \cup Y^- \neq \emptyset$. Since $|\hat{Y}| \leq 1$, there exists an $(X^+ \cup Y^+)$-$(X^- \cup Y^-)$-path in D that omits C. Using Lemma 3.3.(i), it follows that $V(C) \subseteq X$ which contradicts $|V(C) - X| \geq 1$.

Case 1.2. Suppose, without loss of generality, that $X^+ \cup Y^+ = \emptyset$. Using (1) and the assumption that D is strong, it follows that $N^+(C, \hat{Y}) \neq \emptyset$ which immediately implies that $\hat{Y} = \{y\}$ and $Y^- = \emptyset$. Additionally it follows by (1) that $N^-(\hat{X}) - \hat{X} = \emptyset$ and thus, since D is strong, $\hat{X} = \emptyset$. Note that $D[X^-]$ is a tournament and let $D_1, D_2, \ldots, D_q$ be the strong decomposition of $D[X^-]$, where $q \geq 1$. Since $N^-(D_1) - V(D_1) \subseteq \{y\}$, it follows by Lemma 2.5 that $|V(D_1)| \geq 2p - 1$ and by Lemma 2.6 that $|V(D_1)| \geq 2p$ if $y \not\to D_1$.

Case 1.2.1. Suppose that $|V(D_1)| \geq 2p$. Let C' be a Hamiltonian cycle of D_1. Then $V(D) - V(C') - X \neq \emptyset$ and by choice of C it follows that

$$|X| \geq |V(C) \cap X| + |V(C')| \geq 2|V(C')| \geq 4p.$$

Case 1.2.2. Suppose that $|V(D_1)| = 2p - 1$. In this case $y \to D_1$. Then $V(D) - V(C') - X \neq \emptyset$ for an arbitrary Hamiltonian cycle C' of D_1 and thus, we have

$$|V(C) \cap X| \geq |V(C') \cap X| \geq 2p - 1$$

by choice of C. Note that y and each vertex of X^- are adjacent.

If $q \geq 3$ or $|V(D_2)| \geq 3$, we conclude that $|X^-| \geq 2p + 1$ and thus,

$$|X| \geq |V(C) \cap X| + |X^-| \geq 4p.$$

If $q = 2$, $V(D_2) = \{z\}$ and $z \to y$, let P be a Hamiltonian path of D_1 with initial vertex w. In this case $C^* = Pzyw$ is a cycle in D such that $V(D) - V(C^*) - X \neq \emptyset$. By choice of C it follows that

$$|X| \geq |V(C) \cap X| + |X^-| \geq |V(C^*) \cap X| + |X^-| \geq 4p.$$

If $q = 2$, $V(D_2) = \{z\}$ and $y \to z$ or if $q = 1$, it follows that $y \to X^-$ and that $|N^-(y, C)| \geq p$. Let P be a Hamiltonian path of $D[X^-]$. We consider two cases depending on the value of p.

Case 1.2.2.1. Suppose that $p = 2$. Let v_i and v_j be two distinct negative neighbors of y on C. Note that $|V(C) \cap X| \geq |V(D_1)| \geq 2p - 1 = 3$. Therefore we may assume, without loss of generality, that $V(C[v_{j+1}, v_i]) \cap X \neq \emptyset$ and $V(C[v_{i+1}, v_j]) - X \neq \emptyset$. It follows that $C' = yPC[v_{j+1}, v_i]y$ is a cycle in D such that $V(D) - V(C') - X \neq \emptyset$. Consequently,

$$|V(C) \cap X| \geq |V(C') \cap X| \geq |X^-| + 1 \geq 2p,$$

and thus,

$$|X| \geq |V(C) \cap X| + |X^-| \geq 4p - 1.$$

Case 1.2.2.2. Suppose that $p \geq 3$. Let v_i, v_j and v_k be three distinct negative neighbors of y on C such that $i < j < k$. Note that $|V(C) \cap X| \geq |V(D_1)| \geq 2p - 1 \geq 5$. Therefore we may assume, without loss of generality, that $|V(C[v_{j+1}, v_i]) \cap X| \geq 2$ and $V(C[v_{i+1}, v_j]) - X \neq \emptyset$. It follows that $C' = yPC[v_{j+1}, v_i]y$ is a cycle in D such that $V(D) - V(C') - X \neq \emptyset$. Consequently,

$$|V(C) \cap X| \geq |V(C') \cap X| \geq |X^-| + 2 \geq 2p + 1,$$

and thus,

$$|X| \geq |V(C) \cap X| + |X^-| \geq 4p.$$

Case 2. Suppose that $\hat{Y} = \emptyset$. Note that, according to Claim 6, $|Y^+|, |Y^-| \leq 1$ and thus, $|Y^+| = |Y^-| = 1$. Let $Y^+ = \{y_1\}$ and $Y^- = \{y_2\}$. We define

$$A = \{\ v \mid \text{there exists a path from } X^+ \text{ to } v \text{ that omits } y_1 \}$$

and

$$B = \{\ v \mid \text{there exists a path from } v \text{ to } X^- \text{ that omits } y_2 \}.$$

Now we consider two cases.

Case 2.1. Suppose that $X^+ \neq \emptyset$ and $X^- \neq \emptyset$.

Note that, in view of Lemma 3.3, $X^+ \subseteq A \subseteq X - V(C)$, $X^- \subseteq B \subseteq X - V(C)$ and $A \cap B = \emptyset$. Furthermore, $N^+(A) - A \subseteq \{y_1\}$ and $N^-(B) - B \subseteq \{y_2\}$. Using Lemma 2.5, it follows that $|A|, |B| \geq 2p - 1$.

We will show now that $|V(C)| \geq 3$. Assume to the contrary that $|V(C)| = 1$. Since $\delta(D) \geq p \geq 2$, each vertex of A has at least one positive neighbor in A. It follows that $D[A]$ contains a cycle, a contradiction to the choice of D'. This implies that $|V(C)| \geq 3$.

All in all we have shown that

$$|X| \geq |A| + |B| + |V(C)| \geq 4p + 1.$$

Case 2.2. Suppose that $X^+ = \emptyset$ and $X^- \neq \emptyset$. In view of Claim 1 and (1), $N^+(C) - V(C) \subseteq \{y_1\}$. Using Lemma 2.5, it follows that $|V(C)| \geq 2p - 1$.

We will show now that $\hat{X} \neq \emptyset$. Assume to the contrary that $\hat{X} = \emptyset$. Since $\delta \geq p \geq 2$, we conclude that $N^+(y_1, X^-) \neq \emptyset$, a contradiction to Claim 8. It follows that $\hat{X} \neq \emptyset$.

Note that, according to Claims 1 and 4, $N^-(\hat{X}) - \hat{X} \subseteq \{y_1\}$. This implies that $|\hat{X}| \geq 2p - 1$ in view of Lemma 2.5.

Case 2.2.1. Suppose that there exists a vertex $x \in \hat{X}$ such that $y_1 x \notin E(D)$, i.e., $N^-(x) \subseteq \hat{X}$. Using Lemma 2.6, we conclude that $|\hat{X}| \geq 2p$. It follows that

$$|X| \geq |\hat{X}| + |V(C)| + |X^-| \geq 4p.$$

Case 2.2.2. Suppose that $y_1 \to \hat{X}$.

If $N^+(\hat{X}, X^-) \neq \emptyset$, say $x_1 x_2 \in E(D)$, where $x_1 \in \hat{X}$ and $x_2 \in X^-$, then the cycle $C' = C[v_1, v_k] y_1 x_1 x_2 v_1$ yields a contradiction to the choice of D'.

If $N^+(\hat{X}, X^-) = \emptyset$, then note that $N^+(Y^+, X^-) = \emptyset$ in view of Claim 8. We conclude that $N^-(X^-) - X^- \subseteq \{y_2\}$ and thus, using Lemma 2.5, $|X^-| \geq 2p - 1$. It follows that

$$|X| \geq |\hat{X}| + |V(C)| + |X^-| \geq 6p - 3 \geq 4p.$$

The case that $X^+ \neq \emptyset$ and $X^- = \emptyset$ can be solved analogously to Case 2.2.

Case 2.3. Suppose that $X^+ = X^- = \emptyset$. Note that, according to Claim 1, $N^-(\hat{X}, C) = N^+(\hat{X}, C) = \emptyset$. It follows that $N^+(C) - V(C) \subseteq \{y_1\}$ and that $N^-(\hat{X}) - \hat{X} \subseteq \{y_1\}$, in view of Claim 4. Using Lemma 2.5, we conclude that $|V(C)| \geq 2p - 1$ and $|\hat{X}| \geq 2p - 1$ and thus, $|X| = |V(C)| + |\hat{X}| \geq 4p - 2$.

We will show now that if $|X| \leq 4p - 1$, then either $|X| = 4p - 2$, $p \geq 2$ and $D \in \mathcal{T}^*$ or $|X| = 4p - 1$, $p \geq 3$ and $D \in \mathcal{T}^{**}$. Note that, since $C \to y_1$, the set $V(C)$ induces a strong tournament in D.

Assume to the contrary that $\hat{X}$ does not induce a tournament in D. Since D is a local tournament, it follows that $y_1 \not\to \hat{X}$ and $\hat{X} \not\to y_2$. Using Lemma 2.6, it follows that $|\hat{X}| \geq 2p$. Together with the assumption that $|X| \leq 4p - 1$, this implies that $|\hat{X}| = 2p$, $|V(C)| = 2p - 1$ and $|X| = 4p - 1$. Now let x_1 and x_2 be two non-adjacent vertices in $\hat{X}$. Since D is a local tournament, it follows that $N^+(x_1) \cap N^+(x_2) = \emptyset$. But $N^+(x_1) \cup N^+(x_2) \subseteq \{y_2\} \cup (\hat{X} - \{x_1, x_2\})$ and $|\{y_2\} \cup (\hat{X} - \{x_1, x_2\})| = 2p - 1$, a contradiction to the assumption that $\delta(D) \geq p \geq 2$. It follows that $D[\hat{X}]$ is a tournament.

Assume to the contrary that the tournament $T = D[\hat{X}]$ is not strong. We consider the strong decomposition $T_1, T_2, \ldots, T_q$ of T, where $q \geq 2$. Note that $N^+(T_q) - V(T_q) \subseteq \{y_2\}$ and thus, in view of Lemma 2.5, $|V(T_q)| \geq 2p - 1$. Analogously we show that $|V(T_1)| \geq 2p - 1$. It follows that

$$|X| \geq |V(C)| + |V(T_1)| + |V(T_q)| \geq 6p - 3 \geq 4p,$$

a contradiction. It follows that $\hat{X}$ induces a strongly connected tournament.

Assume to the contrary that $|V(C)| < |\hat{X}|$. Since $D[\hat{X}]$ is a strong tournament, it is Hamiltonian, a contradiction to the definition of D'.

It follows that $|\hat{X}| = 2p - 1$ and $2p - 1 \leq |V(C)| \leq 2p$. Using Lemmas 2.5 and 2.6 and the assumption that $\delta(D) \geq p \geq 2$, it follows that $\hat{X}$ induces a $(p-1)$-regular tournament in D. Furthermore, if $|V(C)| = 2p - 1$, then $D[V(C)]$ is a $(p-1)$-regular tournament and if $|V(C)| = 2p$, then $D[V(C)]$ is an almost regular tournament with $p - 1 \leq \delta, \Delta \leq p$.

With $D[\hat{X}] = T_2$, $D[V(C)] = T_1$ ($D[V(C)] = T_0$, if $|V(C)| = 2p$), $u = y_1$ and $v = y_2$, it is easy to see that $D \in \mathcal{T}^*$ ($D \in \mathcal{T}^{**}$, respectively) as defined in Definition 3.1.

Case 3. Suppose that $\hat{Y} \neq \emptyset$. Let $P = z_0 z_1 \dots z_r$ be a shortest path leading from $X^+ \cup Y^+$ to $X^- \cup Y^-$. Then $z_0 \in X^+ \cup Y^+$, $z_r \in X^- \cup Y^-$ and $z_l \in \hat{X} \cup \hat{Y}$ for all $l = 1, 2, \dots, r-1$. In addition $Y \subseteq V(P)$ in view of Lemma 3.3. Since $\hat{Y} \neq \emptyset$, it follows that $r \geq 2$. Let i be the smallest integer such that $z_i \in \hat{Y}$ and let j be the greatest integer such that $z_j \in \hat{Y}$.

First we consider the positive and negative neighborhood of z_1 and z_{r-1}. According to Claims 1, 3, 4 and 5, $N^+(z_1) \subseteq \hat{X} \cup \hat{Y} \cup X^- \cup Y^-$ and $N^-(z_1) \subseteq \hat{X} \cup \hat{Y} \cup X^+ \cup Y^+$. Furthermore, since P is a shortest path, the only positive neighbor of z_1 on P is z_2. Analogously we can show that $N^+(z_{r-1}) \subseteq \hat{X} \cup \hat{Y} \cup X^- \cup Y^-$ and $N^-(z_{r-1}) \subseteq \hat{X} \cup \hat{Y} \cup X^+ \cup Y^+$ and that the only negative neighbor of z_{r-1} on P is z_{r-2}.

Next we consider the positive and negative neighborhood of z_s, where $2 \leq s \leq r-2$. If z_s has a positive neighbor in $X^- \cup Y^-$ or a negative neighbor in $X^+ \cup Y^+$, then P is not a shortest path, a contradiction. It follows that $N^+(z_s) \subseteq \hat{X} \cup \hat{Y}$ and $N^-(z_s) \subseteq \hat{X} \cup \hat{Y}$. We show now that z_{s+1} is the only positive neighbor of z_s on P. Assume to the contrary that $z_t \neq z_{s+1}$ is a positive neighbor of z_s on P. Since P is a shortest path, it follows that $t < s-1$. Using the local tournament property of D, we conclude now that either $z_s \to z_0$ or that there exists a vertex z_q that dominates z_s, where $q \leq s-2$. The first possibility is a contradiction to Claim 4 or Claim 5, respectively, and the second possibility contradicts the minimal choice of P. It follows that $N^+(z_s) \cap V(P) = \{z_{s+1}\}$. We can analogously show that $N^-(z_s) \cap V(P) = \{z_{s-1}\}$.

Next we define

$$A = \{\ v \mid \text{there exists a path from } z_{i-1} \text{ to } v \text{ that omits } z_i\}$$

and

$$B = \{\ v \mid \text{there exists a path from } v \text{ to } z_{j+1} \text{ that omits } z_j\}\,.$$

By definition of A and B, we have $N^+(A) - A \subseteq \{z_i\}$ and $N^-(B) - B \subseteq \{z_j\}$. Using Lemma 3.3 and the observations above, it follows that $A \subseteq \{z_0, z_1, \dots, z_{i-1}\} \cup X^+ \cup Y^+ \cup (\hat{X} - V(P))$ and that $B \subseteq \{z_{j+1}, z_{j+2}, \dots, z_r\} \cup X^- \cup Y^- \cup (\hat{X} - V(P))$. In addition, Lemma 3.3 yields $A \cap B = \emptyset$. Using Lemma 2.5, it follows that $|A|, |B| \geq 2p-1$ and thus $|A \cap X|, |B \cap X| \geq 2p-2$.

Case 3.1. Suppose that $|V(C)| \geq 4$. In this case

$$|X| \geq |A \cap X| + |B \cap X| + |V(C)| \geq 4p$$

and we are done.

Case 3.2. Suppose that $|V(C)| = 3$ and that $|A| \geq 2p$ or $|B| \geq 2p$. Again it follows that

$$|X| \geq |A \cap X| + |B \cap X| + |V(C)| \geq 4p.$$

Case 3.3. Suppose that $|V(C)| = 3$ and $|A| = |B| = 2p-1$. It follows that

$$|X| \geq |A \cap X| + |B \cap X| + |V(C)| \geq 4p-1.$$

It remains to consider the case $p \geq 3$.

We show that $A \to z_i$. Assume to the contrary that $A \not\to z_i$. Then $\delta(D[A]) \geq p-1$ and there exists a vertex $a \in A$ with $d^+(a, D[A]) \geq p$. Using Lemma 2.6, it follows that $|A| \geq 2p$, a contradiction to our assumption. It follows that $A \to z_i$ and thus, since D is a local tournament, A induces a tournament in D.

Let $D_1, D_2, \ldots, D_q$ be the strong decomposition of $D[A]$, where $q \geq 1$. Since D_q is strong and $\delta^+(D_q) \geq p-1$, it has a Hamiltonian cycle of length $2p-1 \geq 5$. This cycle is a contradiction to the choice of D'.

Case 3.4. Suppose that $|V(C)| = 1$. Similarly to Case 3.3, let $D_1, D_2, \ldots, D_q$ be the strong decomposition of the local tournament $D[A]$, where $q \geq 1$. Since D_q is strong and $\delta^+(D_q) \geq p-1 \geq 1$, it has a Hamiltonian cycle of length $2p-1 \geq 3$. This cycle is a contradiction to the choice of D'.

This completes the proof of this lemma. □

Lemma 3.7. *Let D be a strong local tournament and let $C, X^+, X^-, \hat{X}, X, Y^+, Y^-, \hat{Y}, Y$ be defined as above. If $X \subseteq V(C)$, then $V(D) - V(C) = \hat{Y}$ and $|\hat{Y}| = 1$.*

Proof. Using Claim 6, we conclude that $|Y^+|, |Y^-| \leq 1$.

Case 1. Assume that $|Y^+| = |Y^-| = 1$. Since D is strongly connected, there exists an Y^+-Y^--path in D. Let $P = y_0 y_1 \ldots y_s$ be a shortest such path. According to Lemma 3.3, $V(P) = Y$. Since P is minimal, it follows that $N^+(y_0) = \{y_1\}$ which contradicts $\delta^+(D) \geq 2$.

Case 2. Assume that $|Y^+| = 1$ and $Y^- = \emptyset$. Since D is strong and $C \to Y^+$, we conclude that $\hat{Y} \neq \emptyset$. Using Claim 3, it follows that $N^+(\hat{Y}, C) = N^-(\hat{Y}, C) = \emptyset$. It follows $N^-(C) \subseteq V(C)$, a contradiction to the assumption that D is strong.

The case that $|Y^-| = 1$ and $Y^+ = \emptyset$ can be solved analogously to Case 2.

Case 3. Assume that $Y^+ = Y^- = \emptyset$. Then $\hat{Y} \neq \emptyset$. Since D is strong, $N^+(\hat{Y}, C), N^-(\hat{Y}, C) \neq \emptyset$. Using Claim 3, it follows that $|\hat{Y}| \leq 1$ which completes the proof of this lemma. □

Theorem 3.8. *Let $p \geq 2$ be an integer and let D be a strongly connected local tournament with $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p-3$ and $X \neq V(D)$, then there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$.*

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p-3$ and $X \neq V(D)$. Let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above.

Assume that $X - V(C) \neq \emptyset$. If $|X| = |V(D)| - 1$, then we get a contradiction by Lemma 3.5. If $|X| \leq |V(D)| - 2$, then we get a contradiction by Lemma 3.6.(a).

Hence, we obtain $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ by Lemma 3.7. □

Theorem 3.9. *Let $p \geq 2$ be an integer and let D be a strongly connected local tournament with $|V(D)| \geq 4p$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p-2$, then either*

(i) there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or

(ii) $D \in \mathcal{T}^*$.

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p-2$ and let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above. In addition, let $D \notin \mathcal{T}^*$.

Assume that $X - V(C) \neq \emptyset$. Since $D \notin \mathcal{T}^*$, Lemma 3.6.(a) implies that $|X| \geq 4p-1$, a contradiction to our assumption. Hence, by Lemma 3.7, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

Theorem 3.10. *Let $p \geq 3$ be an integer. Let D be a strongly connected local tournament with $|V(D)| \geq 4p+1$ and $\delta(D) \geq p$. If $X \subseteq V(D)$ with $|X| \leq 4p-1$, then either*

(i) there exists a cycle C in D such that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$ or

(ii) $D \in \mathcal{T}^{**}$.

Proof. Let X be a subset of $V(D)$ such that $|X| \leq 4p-1$ and let C, X^+, X^-, $\hat{X}$, X, Y^+, Y^-, $\hat{Y}$ and Y be defined as above. In addition, let $D \notin \mathcal{T}^{**}$.

Assume that $X - V(C) \neq \emptyset$. Note that $D \notin \mathcal{T}^*$, since $|V(D)| \geq 4p+1$. Additionally $D \notin \mathcal{T}^{**}$ and thus, Lemma 3.6.(b) implies that $|X| \geq 4p$, a contradiction to our assumption. Hence, by Lemma 3.7, it follows that $X \subseteq V(C)$ and $|V(C)| = |V(D)| - 1$. □

Combining Lemma 2.4 and the theorems above we derive the following results.

Theorem 3.11. *Let D be a strong local tournament and let $p \geq 2$ be an integer. If $\delta(D) \geq p$, then D has at least $k = \min\{|V(D)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 3.12. *Let D be a strong local tournament such that $D \notin \mathcal{T}^*$ and let $p \geq 2$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p$, then D has at least $k = 4p-1$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Theorem 3.13. *Let D be a strong local tournament such that $D \notin \mathcal{T}^{**}$ and let $p \geq 3$ be an integer. If $\delta(D) \geq p$ and $|V(D)| \geq 4p+1$, then D has at least $k = 4p$ vertices $x_1, x_2, \ldots, x_k$ such that $D - x_i$ is strong for $i = 1, 2, \ldots, k$.*

Since the classes $\mathcal{T}^*$ and $\mathcal{T}^{**}$ of local tournaments do not contain any tournaments, Theorems 1.6-1.8 by Kotani [7] are direct consequences of Theorems 3.11-3.13.

Acknowledgement. The authors would like to thank the anonymous referee for many suggestions that helped the presentation of the article.

References

[1] J. Bang-Jensen: Locally semicomplete digraphs: a generalization of tournaments, *J. Graph Theory* **14** (1990), 371-390.

[2] J. Bang-Jensen and G. Gutin: Generalizations of tournaments: a survey, *J. Graph Theory* **28** (1998), 171-202.

[3] Y. Guo: Locally semicomplete digraphs, *Ph.D. thesis, RWTH Aachen, Aachener Beiträge zur Mathematik* **13** (1995), 92 p.

[4] Y. Guo and L. Volkmann: Connectivity properties of locally semicomplete digraphs, *J. Graph Theory* **18** (1994), 269-280.

[5] J. Huang: Tournament-Like Oriented Graphs, *Ph.D. thesis, Simon Fraser University* (1992).

[6] G. Korvin: Some combinatorial problems on complete directed graphs, *Theory of Graphs* (Internat. Sympos., Rome, 1966) pp. 197-213, Gordon and Breach, New York; Dunod, Paris 1967.

[7] K. Kotani: Nonseparating vertices in tournaments with large minimum degree, *Australas. J. Combin.* **36** (2006), 27-38.

[8] M. Las Vergnas: Sur le nombre de circuits dans un tournoi fortement connexe, *Cahiers Centre Études Recherche Opér.* **17** (1975), no. 2-3-4, 261-265.

[9] J.W. Moon: On subtournaments of a tournament, *Canad. Math. Bull.* **9** (1966), 297-301.