

# A lower bound for the distance $k$ -domination number of trees

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## Abstract

A subset  $D$  of vertices of a graph  $G = (V, E)$  is a distance  $k$ -dominating set for  $G$  if the distance between every vertex of  $V - D$  and  $D$  is at most  $k$ . The minimum size of a distance  $k$ -dominating set of  $G$  is called the distance  $k$ -domination number  $\gamma_k(G)$  of  $G$ . In this paper we prove that  $(2k + 1)\gamma_k(T) \geq |V| + 2k - kn_1$  for each tree  $T = (V, E)$  with  $n_1$  leafs, and we characterize the class of trees that satisfy the equality  $(2k + 1)\gamma_k(T) = |V| + 2k - kn_1$ . Our results generalize those of Lemanska [4] for  $k = 1$  and of Cyman, Lemanska and Raczek [1] for  $k = 2$ .

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## 1 Terminology and introduction

In this paper we consider finite, undirected and simple graphs  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ . The number of vertices  $|V|$  is called the *order* of  $G$  and is denoted by  $n(G)$ . For two distinct vertices  $u$  and  $v$  the *distance*  $d(u, v)$  between  $u$  and  $v$  is the length of a shortest path between  $u$  and  $v$ . The *diameter* of a graph  $G$  is  $\text{diam}(G) = \max \{d(u, v) \mid u, v \in V(G)\}$ . If  $X$  and  $Y$  are two disjoint subsets of  $V$ , then the distance between  $X$  and  $Y$  is defined as  $d(X, Y) = \min \{d(x, y) \mid x \in X, y \in Y\}$ . The *open  $k$ -neighborhood*  $N^k(X)$  of a subset  $X \subseteq V$  is the set of vertices in  $V \setminus X$  of distance at most  $k$  from  $X$  and the *closed  $k$ -neighborhood* is defined by  $N^k[X] = N^k(X) \cup X$ . If  $X = \{v\}$  is a single vertex, then we denote the (closed)  $k$ -neighborhood of  $v$  by  $N^k(v)$  ( $N^k[v]$ , respectively). The (closed) 1-neighborhood of a vertex  $v$  or a set  $X$  of vertices is usually denoted by  $N(v)$  or  $N(X)$ , respectively ( $N[v]$  or  $N[X]$ , respectively). Now let  $U$  be an arbitrary subset of  $V$  and  $u \in U$ . We say that  $v$  is a *private  $k$ -neighbor of  $u$  with respect to  $U$*  if  $d(u, v) \leq k$  and  $d(u', v) > k$  for all  $u' \in U - \{u\}$ , that is  $v \in N^k[u] - N^k[U - \{u\}]$ . The *private  $k$ -neighborhood of  $u$  with respect to  $U$*  will be denoted by  $PN^k[u, U]$ .

For a vertex  $v \in V$  we define the *degree* of  $v$  as  $d(v) = |N(v)|$ . A vertex of degree one is called a *leaf* and the set of leafs of a graph  $G$  is denoted by  $\Omega(G)$ . In addition, we define  $\Omega_l(G)$  to be the set of leafs belonging to a longest path of  $G$ . The number of leafs  $|\Omega(G)|$  will be denoted by  $n_1(G)$ .

For a tree  $T$  and an edge  $uv \in E(T)$ , let  $T_u$  and  $T_v$  denote the components of  $T - uv$  to which the vertices  $u$  and  $v$  belong, respectively.

A *star*  $K_{1,t}$  is a complete bipartite graph with partition sets  $X, Y$  such that  $|X| = 1$  and  $|Y| = t$ . The *edge subdivision* in a graph  $G$  is the following operation; remove one edge  $e = uv$  of  $G$  and add a new vertex  $w$  and the edges  $uw$  and  $vw$ . A *k-times subdivided star*  $SS_t^k$  is obtained from a star  $K_{1,t}$  by subdividing each edge by exactly  $k$  vertices.

A set  $D \subseteq V$  is said to be a *distance k-dominating set* of  $G$  if the distance  $d(u, D)$  between each vertex  $u \in V - D$  and  $D$  is at most  $k$ . The minimum cardinality of a distance  $k$ -dominating set in  $G$  is the *distance k-domination number* of  $G$ , denoted by  $\gamma_k(G)$ . Note that the distance 1-domination number  $\gamma_1(G)$  is the usual *domination number*  $\gamma(G)$ .

In 1975, Meir and Moon [5] introduced the concept of a  $k$ -dominating set (called a ' $k$ -covering' in [5]) in a graph, and established an upper bound for the  $k$ -domination number of a tree. More precisely, they proved that  $\gamma_k(T) \leq |V(T)|/(k+1)$  for every tree  $T$ . This leads immediately to  $\gamma_k(G) \leq |V(G)|/(k+1)$  for an arbitrary graph  $G$ . In 1991, Topp and Volkmann [6] gave a complete characterization of the class of graphs  $G$  that fulfill the equality  $\gamma_k(G) = |V(G)|/(k+1)$ . For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [2], [3].

In this paper we establish a lower bound for the distance  $k$ -domination number of trees. More precisely, we prove that  $(2k+1)\gamma_k(T) \geq |V| + 2k - kn_1(T)$  for each tree  $T = (V, E)$  with  $n_1(T)$  leaves, and we characterize the class of trees that satisfy the equality  $(2k+1)\gamma_k(T) = |V| + 2k - kn_1(T)$ . Our results generalize those of Lemanska [4] for  $k = 1$  and of Cyman, Lemanska and Raczek [1] for  $k = 2$ .

## 2 Results

The following lemma is a preparation for Theorem 2.2.

**Lemma 2.1** *Let  $T$  be a tree with  $\gamma_k(T) > 1$ . Then there exists an edge  $uv$  in  $T$  such that  $\gamma_k(T) = \gamma_k(T_u) + \gamma_k(T_v)$ .*

**Proof.** Let  $P = v_0v_1 \dots v_l$  be a longest path in  $T$ . Since  $\gamma_k(T) > 1$ , we have  $l \geq 2k + 1$ . Now let  $D$  be a minimum distance  $k$ -dominating set of  $T$  such that

- (1)  $v_k \in D$  and
- (2)  $\sum_{x \in D} d(x, P)$  is minimal.

For  $1 \leq i \leq l - 1$  let  $T_i$  be the component of  $T - \{v_{i-1}v_i, v_iv_{i+1}\}$  that contains the vertex  $v_i$ . Note that condition (2) implies that all vertices  $x \in V(T_i) \cap D$  satisfy the inequality  $d(v_i, x) \leq i - k$  for  $i \geq k$ . Let  $0 \leq p \leq k$  be the greatest integer such that  $v_k$  has at least one private  $k$ -neighbor in  $T_{k+p}$ . We will now show that  $d(v_{k+p}, v) \leq k - p$  for all vertices  $v \in V(T_{k+p})$ , i.e.,  $V(T_{k+p}) \subseteq N^k[v_k]$ . Let  $y \in PN^k[v_k, D] \cap V(T_{k+p})$  be a private  $k$ -neighbor of  $v_k$  in  $T_{k+p}$  and suppose that  $z \in V(T_{k+p}) - N^k[v_k]$  is not a  $k$ -neighbor of  $v_k$ . Then  $d(v_{k+p}, y) \leq k - p$  and  $k - p + 1 \leq d(v_{k+p}, z) \leq k + p$  (the latter inequality holds because  $P$  is a longest path in  $T$ ). In addition, there exists a vertex  $v_k \neq x \in D$  such that  $z \in N^k[x]$ . We shall show now that  $y$  is also a  $k$ -neighbor of  $x$ .

Suppose first that  $x \in V(T_{k+p})$ . Since  $d(x, v_{k+p}) \stackrel{(2)}{\leq} p$  and  $d(v_{k+p}, y) \leq k - p$ , it follows that  $d(x, y) \leq d(x, v_{k+p}) + d(v_{k+p}, y) \leq k$ , a contradiction.

Suppose second that  $x \notin V(T_{k+p})$ , i.e.,  $x \in V(T_j)$  for an integer  $1 \leq j \leq l - 1$ . Then  $d(x, y) = d(x, v_j) + d(v_j, v_{k+p}) + d(v_{k+p}, y) \leq d(x, v_j) + d(v_j, v_{k+p}) + d(v_{k+p}, z) = d(x, z)$  and thus, since  $z \in N^k[x]$ , we conclude that  $y \in N^k[x]$ , a contradiction.

Let us now remove the edge  $uv = v_{k+p}v_{k+p+1}$ . We shall show now that  $\{v_k\}$  is a distance  $k$ -dominating set of  $T_u$  and that  $D - v_k$  is a distance  $k$ -dominating set of  $T_v$  which completes the proof of this lemma.

Since  $v_k$  has no private  $k$ -neighbor in  $T_v$ , it is immediate that  $D - v_k$  is a distance  $k$ -dominating set of  $T_v$ .

Now assume that there exists a vertex  $y \in V(T_u)$  that is no  $k$ -neighbor of  $v_k$ . Then  $y \in V(T_{k+q})$  for an integer  $1 \leq q \leq p - 1$  and  $d(v_{k+q}, y) \geq k - q + 1$ . Let  $v_k \neq x \in D$  be a  $k$ -neighbor of  $y$ . We shall now conclude a contradiction to the assumption that  $v_k$  has a private  $k$ -neighbor in  $T_{k+p}$ . Let  $x \in V(T_{k+j})$  for an integer  $1 \leq j \leq l - 1$  and let  $z$  be an arbitrary vertex of  $T_{k+p}$ . Then

$$d(x, z) \leq d(x, v_{k+j}) + d(v_{k+j}, v_{k+p}) + d(v_{k+p}, z) \stackrel{(2)}{\leq} j + (p - j) + (k - p) = k,$$

a contradiction.  $\square$

Using Lemma 2.1 in the induction step, we will now establish the following lower bound on the distance  $k$ -domination number.

**Theorem 2.2** *Let  $T$  be a tree with  $n_1(T)$  leaves. Then*

$$kn_1(T) \geq n(T) + 2k - (2k + 1)\gamma_k(T).$$

**Proof.** We use induction on the distance  $k$ -domination number  $\gamma_k(T)$ . If  $\gamma_k(T) = 1$ , then  $\text{diam}(T) \leq 2k$  and consequently,  $kn_1(T) \geq n(T) - 1$ . Hence, the statement holds for every tree  $T$  with  $\gamma_k(T) = 1$ .

Now let  $T$  be a tree with  $\gamma_k(T) \geq 2$  and assume that  $kn_1(T') \geq n(T') + 2k - (2k + 1)\gamma_k(T')$  for every tree  $T'$  with  $\gamma_k(T') < \gamma_k(T)$ .

By Lemma 2.1, there exists an edge  $uv$  in  $T$  such that  $\gamma_k(T) = \gamma_k(T_u) + \gamma_k(T_v)$ . By induction we have  $kn_1(T_u) \geq n(T_u) + 2k - (2k + 1)\gamma_k(T_u)$  and  $kn_1(T_v) \geq n(T_v) + 2k - (2k + 1)\gamma_k(T_v)$ . It follows that  $kn_1(T_u) + kn_1(T_v) \geq n(T_u) + n(T_v) + 4k - (2k + 1)(\gamma_k(T_u) + \gamma_k(T_v))$ . Furthermore, it is easy to see that  $n_1(T_u) + n_1(T_v) \leq n_1(T) + 2$  which finally implies that  $kn_1(T) \geq n(T) + 2k - (2k + 1)\gamma_k(T)$ .  $\square$

Now we will characterize the class of trees  $T$  for which the bound established in Theorem 2.2 is sharp, i.e.,  $kn_1(T) = n(T) + 2k - (2k + 1)\gamma_k(T)$ .

Let  $\mathcal{R}$  denote the family of trees in which the distance between each pair of distinct leaves is congruent  $2k$  modulo  $(2k + 1)$ , that is

$$\mathcal{R} = \{T \mid T \text{ is a tree and } d(x, y) \equiv 2k \pmod{(2k + 1)} \text{ for all distinct vertices } x, y \in \Omega(T)\}.$$

The following lemma is a preparatory result for Theorem 2.4.

**Lemma 2.3** *If  $T$  is a tree belonging to the family  $\mathcal{R}$  and  $\gamma_k(T) > 1$ , then there exists an edge  $uv$  in  $T$  such that both  $T_u$  and  $T_v$  belong to  $\mathcal{R}$ ,  $\gamma_k(T) = \gamma_k(T_u) + \gamma_k(T_v)$  and  $n_1(T) = n_1(T_u) + n_1(T_v) - 2$ .*

**Proof.** Let  $T \in \mathcal{R}$  with  $\gamma_k(T) \geq 2$  and let  $P = v_0v_1 \dots v_l$  be a longest path in  $T$ . In addition, let  $D$  be a minimum distance  $k$ -dominating set of  $T$  containing the vertex  $v_k$ . Then  $l \equiv 2k \pmod{2k+1}$ ,  $l \geq 4k+1$  and  $v_k \in D$ .

We will now show that  $d(v_{k+1}) = d(v_{k+2}) = \dots = d(v_{3k}) = 2$ . Suppose to the contrary that  $N(v_i) - V(P) \neq \emptyset$  for some  $i \in \{k+1, k+2, \dots, 3k\}$ . Then there exists an endvertex  $y \in \Omega(T)$  such that  $d(y, v_i) = d(y, P) > 0$ . In order to derive a contradiction, we will now compute the possible values for  $i$ . We have  $d(y, v_i) = d(y, v_0) - d(v_i, v_0) = d(y, v_0) - i$  and  $d(v_i, v_l) = d(v_0, v_l) - d(v_0, v_i) = d(v_0, v_l) - i$ . It follows that  $d(y, v_l) = d(y, v_i) + d(v_i, v_l) = d(y, v_0) + d(v_0, v_l) - 2i$ . Since  $v_0, v_l$  and  $y$  are endvertices and  $T \in \mathcal{R}$ , it follows that  $2i \equiv 2k \pmod{2k+1}$ . The latter together with  $k+1 \leq i \leq 3k$  leads immediately to a contradiction.

It follows that  $d(v_{k+1}) = d(v_{k+2}) = \dots = d(v_{3k}) = 2$  which means we can choose  $D$  such that  $v_{3k+1} \in D$ . Let us remove the edge  $uv = v_{2k}v_{2k+1}$  from  $T$ . Then  $n_1(T) = n_1(T_u) + n_1(T_v) - 2$ ,  $\gamma_k(T_u) = 1$  and  $D - v_k$  is a minimum distance  $k$ -dominating set of  $T_v$ . Thus,  $\gamma_k(T_u) + \gamma_k(T_v) = \gamma_k(T)$ . Since  $T_u = SS_t^{k-1}$  is a star with all edges  $(k-1)$ -times subdivided,  $T_u \in \mathcal{R}$ . As  $T \in \mathcal{R}$ , we have  $d(v_0, x) = 2k \pmod{2k+1}$  for every vertex  $v_0 \neq x \in \Omega(T)$ . Since  $d(v_0, v_{2k+1}) = 2k+1$ , we obtain  $d(v_{2k+1}, x) = 2k \pmod{2k+1}$  for every vertex  $v_{2k+1} \neq y \in \Omega(T_v)$  and consequently,  $T_v \in \mathcal{R}$ . This completes the proof of this lemma.  $\square$

A look at the proof of Lemma 2.3 shows that  $T_u$  can actually be chosen as a star  $SS_t^{k-1}$  with each edge  $(k-1)$ -times subdivided and thus,  $\gamma_k(T_u) = 1$  and  $\gamma_k(T_v) = \gamma_k(T) - 1$ . The equality  $n_1(T) = n_1(T_u) + n_1(T_v) - 2$  means that  $uv$  is an edge that joins the endvertices  $u$  and  $v$  of  $T_u$  and  $T_v$ , respectively. By applying Lemma 2.3 repeatedly, we can now characterize the trees  $T \in \mathcal{R}$  with  $\gamma_k(T) = r$  as follows. If  $T$  is a tree in  $\mathcal{R}$  with  $\gamma_k(T) = r$ , then  $T$  arises from  $r$  disjoint trees  $SS_{t_j}^{k-1}$  ( $j = 1, 2, \dots, r$ ) by adding edges that join endvertices of distinct trees.

Using Lemma 2.3, we will now characterize the class of trees  $T$  which fulfill the equality  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$ .

**Theorem 2.4** *Let  $T$  be a tree with  $n_1(T)$  leaves. Then  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$  if and only if  $T$  belongs to the family  $\mathcal{R}$ .*

**Proof.** Suppose first that  $T \in \mathcal{R}$ . If  $\gamma_k(T) = 1$ , then  $T = SS_t^{k-1}$  is a star with each edge  $(k-1)$ -times subdivided and  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$  is obvious. Assume now that  $\gamma_k(T) \geq 2$  and that  $kn_1(T') = n(T') + 2k - (2k+1)\gamma_k(T')$  for every tree  $T' \in \mathcal{R}$  with  $\gamma_k(T') < \gamma_k(T)$ .

According to Lemma 2.3, there exists an edge  $uv$  in  $T$  such that  $T_u, T_v \in \mathcal{R}$ ,  $\gamma_k(T) = \gamma_k(T_u) + \gamma_k(T_v)$  and  $n_1(T) = n_1(T_u) + n_1(T_v) - 2$ . By the induction hypothesis,  $kn_1(T_u) = n(T_u) + 2k - (2k+1)\gamma_k(T_u)$  and  $kn_1(T_v) = n(T_v) + 2k - (2k+1)\gamma_k(T_v)$ . By adding these equalities we finally conclude that  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$ .

Suppose second that  $T$  fulfills the equality  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$ . If  $\gamma_k(T) = 1$ , then the equality yields  $kn_1(T) = n(T) - 1$ . This together with  $\text{diam}(T) \leq 2k$

implies that  $T = SS_t^{k-1}$  is a star with each edge  $(k-1)$ -times subdivided and  $T \in \mathcal{R}$  is obvious. Now let  $T$  be a tree with  $\gamma_k(T) > 1$  that fulfills the equality  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$  and assume that  $T' \in \mathcal{R}$  for all trees  $T'$  with  $\gamma_k(T') < \gamma_k(T)$  and  $kn_1(T') = n(T') + 2k - (2k+1)\gamma_k(T')$ . According to Lemma 2.1 there exists an edge  $uv$  in  $T$  such that  $\gamma_k(T) = \gamma_k(T_u) + \gamma_k(T_v)$ . Since  $kn_1(T) = n(T) + 2k - (2k+1)\gamma_k(T)$ , it follows that  $n_1(T) = n_1(T_u) + n_1(T_v) - 2$ ,  $kn_1(T_u) = n(T_u) + 2k - (2k+1)\gamma_k(T_u)$  and  $kn_1(T_v) = n(T_v) + 2k - (2k+1)\gamma_k(T_v)$ . Note that this means that  $T$  arises from  $T_u$  and  $T_v$  by adding the edge  $uv$  which joins the endvertices  $u$  and  $v$  of  $T_u$  and  $T_v$ , respectively. In addition, we conclude that  $T_u, T_v \in \mathcal{R}$  by the induction hypothesis. The latter together with the observation before implies that  $T \in \mathcal{R}$  which completes the proof of this theorem.  $\square$

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