The $\{k\}$-domatic number of a graph

[1]D. Meierling, [2]S.M. Sheikholeslami and [1]L. Volkmann

[1]Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
`meierling,volkm@math2.rwth-aachen.de`

[2]Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
`s.m.sheikholeslami@azaruniv.edu`

Abstract

For a positive integer k, a $\{k\}$-*dominating function* of a graph G is a function f from the vertex set $V(G)$ to the set $\{0, 1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u \in N[v]} f(u) \geq k$ is fulfilled, where $N[v]$ is the closed neighborhood of v. The $\{1\}$-dominating function is the same as the ordinary domination. A set $\{f_1, f_2, \ldots, f_d\}$ of $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a $\{k\}$-*dominating family* (of functions) on G. The maximum number of functions in a $\{k\}$-dominating family on G is the $\{k\}$-*domatic number* of G, denoted by $d_{\{k\}}(G)$. Note that $d_{\{1\}}(G)$ is the classical domatic number $d(G)$. In this paper we initiate the study of the $\{k\}$-domatic number in graphs and we present some bounds for $d_{\{k\}}(G)$. Many of the known bounds of $d(G)$ are immediate consequences of our results.

Keywords: $\{k\}$-dominating function, $\{k\}$-domination number, $\{k\}$-domatic number.
MSC 2000: 05C69

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Consult [5, 7] for the notation and terminology which are not defined here.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A domatic partition is a partition of V into dominating sets, and the domatic number $d(G)$ is the largest number of sets in a domatic partition. The domatic number was introduced by Cockayne and Hedetniemi [3]. In their paper, they showed that

$$\gamma(G) \cdot d(G) \leq n. \tag{1}$$

For a positive integer k, a $\{k\}$-*dominating function* ($\{k\}$DF) of a graph G is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u\in N[v]} f(u) \geq k$ is fulfilled. The *weight* of a $\{k\}$DF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The $\{k\}$-*domination number* of a graph G, denoted by $\gamma_{\{k\}}(G)$, is the minimum weight of a $\{k\}$DF of G. A $\gamma_{\{k\}}(G)$-*function* is a $\{k\}$-dominating function of G with weight $\gamma_{\{k\}}(G)$. Note that $\gamma_{\{1\}}(G)$ is the classical domination number $\gamma(G)$. The $\{k\}$-domination number was introduced by G.S. Domke et al. [4] and has been studied by several authors (see for example [4, 6]). Furthermore, it was proved in [4] that

$$\gamma_{\{k\}}(G) \leq k\gamma(G)$$

for any $k \geq 2$ and any graph G.

A set $\{f_1, f_2, \ldots, f_d\}$ of $\{k\}$-dominating functions of G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a $\{k\}$-*dominating family* (of functions) on G. The maximum number of functions in a $\{k\}$-dominating family ($\{k\}$D family) on G is the $\{k\}$-*domatic number* of G, denoted by $d_{\{k\}}(G)$. The $\{k\}$-domatic number is well-defined and

$$d_{\{k\}}(G) \geq 1 \tag{2}$$

for all graphs G, since the set consisting of the function $f : V(G) \rightarrow \{0,1,2,\ldots,k\}$ defined by $f(v) = k$ for each $v \in V(G)$, forms a $\{k\}$D family on G.

Our purpose in this paper is to initiate the study of the $\{k\}$-domatic number in graphs. We first study basic properties and bounds for the $\{k\}$-domatic number of a graph. In addition, we determine the $\{k\}$-domatic number of some classes of graphs.

We start with the following observations. Zelinka showed in [8] that $d_k \leq d_m$ whenever $k \leq m$.

Observation 1. If G is a graph, then $d_{\{k\}}(G) \geq d(G)$.

Proof. If $\{f_1, f_2 \ldots, f_{d(G)}\}$ is a family of dominating functions of G, then $\{kf_1, kf_2 \ldots, kf_{d(G)}\}$ is a $\{k\}$D family of G. □

Observation 2. For each two positive integers n, k, $d_{\{k\}}(\overline{K_n}) = 1$ and $d_{\{k\}}(K_n) = n$.

Observation 3. Let G be a connected graph of order $n \geq 2$, and let k be a positive integer. Then $d_{\{k\}}(G) \geq 2$.

Proof. For a fixed vertex $v \in V(G)$, let $V_i = \{u \in V(G) \mid d_G(u,v) = i\}$ for $i = 0,1,\ldots,h$, where h is the eccentricity of v. Define $f : V(G) \rightarrow \{1,2,\ldots,k\}$ by $f(u) = 0$ for $u \in V_i$ when i is even and $f(u) = k$ otherwise. Also define $g : V(G) \rightarrow \{1,2,\ldots,k\}$ by $g(u) = k$ for $u \in V_i$ when i is even and $f(u) = 0$ otherwise. Obviously, $\{f, g\}$ is a $\{k\}$D family on G and the result follows. □

Corollary 4. If k is a positive integer and G a graph without isolated vertices, then $d_{\{k\}}(G) \geq 2$.

Observation 5. Let G be a Hamiltonian graph of order $n \equiv 0 \pmod 3$, and let k be a positive integer. Then $d_{\{k\}}(G) \geq 3$.

Proof. Let $n = 3t$ for a positive integer t, and let $x_1x_2\ldots x_{3t}x_1$ be a Hamiltonian cycle of G. Define $f_j : V(G) \rightarrow \{1,2,\ldots,k\}$ by $f_j(x_i) = k$ for $i \equiv j \pmod 3$ and $f_j(x_i) = 0$ otherwise for $j \in \{0,1,2\}$. Now it is easy to see that $\{f_1, f_2, f_3\}$ is a $\{k\}$D family on G, and the proof is complete. □

2 Properties of the $\{k\}$-domatic number

In this section we mainly present basic properties of $d_{\{k\}}(G)$ and bounds on the $\{k\}$-domatic number of a graph.

Theorem 6. If $k \geq 1$ is an integer and G a graph of order n, then

$$\gamma_{\{k\}}(G) \cdot d_{\{k\}}(G) \leq kn.$$

Moreover, if $\gamma_{\{k\}}(G) \cdot d_{\{k\}}(G) = kn$, then for each $\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_{\{k\}}(G)$, each function f_i is a $\gamma_{\{k\}}(G)$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a $\{k\}$D family on G such that $d = d_{\{k\}}(G)$. Then

$$\begin{aligned} d \cdot \gamma_{\{k\}}(G) &= \sum_{i=1}^{d} \gamma_{\{k\}}(G) \leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V} k = kn. \end{aligned}$$

If $\gamma_{\{k\}}(G) \cdot d_{\{k\}}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the $\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{\{k\}}(G)$. Thus each function f_i is a $\gamma_{\{k\}}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V$. □

The case $k = 1$ in Theorem 6 leads to the well-known inequality $\gamma(G) \cdot d(G) \leq n$, given by Cockayne and Hedetniemi [3] in 1977.

Theorem 7. If k is a positive integer and G a graph of order n, then

$$d_{\{k\}}(G) \leq n,$$

with equality if and only if G is isomorphic to the complete graph K_n.

Proof. Since $\gamma_{\{k\}}(G) \geq k$, it follows from Theorem 6 that

$$d_{\{k\}}(G) \leq \frac{kn}{\gamma_{\{k\}}(G)} \leq \frac{kn}{k} = n,$$

and this is the desired inequality.

If G is isomorphic to the complete graph K_n then by Observation 2, $d_{\{k\}}(G) = n$.

Conversely, let $d_{\{k\}}(G) = n$. It follows from Theorem 6 that $\gamma_{\{k\}}(G) = k$. Let $\{f_1, f_2, \ldots, f_n\}$ be a $\{k\}$D family on G. Theorem 6 implies that $\sum_{v \in V(G)} f_i(v) = k$ for each i and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V$. Let $v \in V(G)$ be an arbitrary vertex of G. Then $f_i(v) > 0$ for some i because $\sum_{i=1}^{d} f_i(v) = k$. Since $f_i(v) > 0$ and

$$k = \sum_{v \in V(G)} f_i(v) \geq \sum_{x \in N[u]} f_i(x) \geq k$$

for each $u \in V(G)$, we deduced that $v \in N[u]$. Thus v is adjacent to all vertices of G. Since v is an arbitrary vertex of G, the result follows. □

Theorem 8. If $k \geq 1$ is an integer and G a graph of order n, then

$$\gamma_{\{k\}}(G) + d_{\{k\}}(G) \leq nk + 1,$$

with equality if and only if G is isomorphic to $\overline{K_n}$ or to K_n when $k = 1$.

Proof. Applying Theorem 6, we obtain

$$\gamma_{\{k\}}(G) + d_{\{k\}}(G) \leq \frac{kn}{d_{\{k\}}(G)} + d_{\{k\}}(G). \tag{3}$$

Note that $d_{\{k\}}(G) \geq 1$, by inequality (2), and that Theorem 7 implies that $d_{\{k\}}(G) \leq n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $1 \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n$, we obtain the desired bound as follows

$$\gamma_{\{k\}}(G) + d_{\{k\}}(G) \leq \max\left\{kn + 1, \frac{kn}{n} + n\right\} = nk + 1.$$

If G is isomorphic to $\overline{K_n}$, then $\gamma_{\{k\}}(G) = kn$ and $d_{\{k\}}(G) = 1$ and thus $\gamma_{\{k\}}(G) + d_{\{k\}}(G) = nk + 1$. If $G = K_n$ and $k = 1$, then $\gamma(G) + d(G) = n + 1$.

Conversely, assume that G has a component H of order at least 2 and that $G \neq K_n$ when $k = 1$. Choose a vertex $w \in V(H)$, and define $f : V(G) \to \{1, 2, \ldots, k\}$ by $f(w) = 0$ and $f(x) = k$ for $x \in V(G) \setminus \{w\}$. Then f is a $\{k\}$-dominating function on G and hence $\gamma_{\{k\}}(G) \leq k(n-1)$. If G contains an isolated vertex, then $d_{\{k\}}(G) = 1$ and consequently $\gamma_{\{k\}}(G) + d_{\{k\}}(G) \leq k(n-1) + 1 \leq kn$.

If G is without isolated vertices, then Corollary 4 implies that $d_{\{k\}}(G) \geq 2$. Using this fact and inequality (3), we obtain for $k \geq 2$

$$\begin{aligned}
\gamma_{\{k\}}(G) + d_{\{k\}}(G) &\leq \frac{kn}{d_{\{k\}}(G)} + d_{\{k\}}(G) \\
&\leq \max\left\{\frac{kn}{2} + 2, \frac{kn}{n} + n\right\} \\
&= \max\left\{\frac{kn}{2} + 2, k + n\right\} \\
&= \frac{kn}{2} + 2 \leq kn.
\end{aligned}$$

Finally assume that $k = 1$, $G \neq K_n$ and G is without isolated vertices. Then $n \geq 3$ and $\gamma(G) \leq n - 1$. If $\gamma(G) = 1$, then it follows from Theorem 7 that $d(G) \leq n - 1$ and thus $\gamma(G) + d(G) \leq n$.

If $\gamma(G) \geq 2$, then Theorem 6 and $\gamma(G) \leq n - 1$ imply that

$$\begin{aligned}
\gamma(G) + d(G) &\leq \gamma(G) + \frac{n}{\gamma(G)} \\
&\leq \max\left\{2 + \frac{n}{2}, n - 1 + \frac{n}{n-1}\right\} \\
&= n - 1 + \frac{n}{n-1} < n + 1,
\end{aligned}$$

and thus $\gamma(G) + d(G) \leq n$ when $\gamma(G) \geq 2$. This completes the proof. □

Corollary 9 (Cockayne and Hedetniemi [3] 1977)**.** If G is a graph of order $n \geq 1$, then $\gamma(G) + d(G) \leq n + 1$, with equality if and only if G is isomorphic to K_n or to $\overline{K_n}$.

The proof of Theorem 8 leads to the next result immediately.

Theorem 10. If $k \geq 2$ is an integer and G a graph without isolated vertices of order n, then

$$\gamma_{\{k\}}(G) + d_{\{k\}}(G) \leq \frac{kn}{2} + 2.$$

Theorem 11. For every graph G,

$$d_{\{k\}}(G) \leq \delta(G) + 1.$$

Moreover, if $d_{\{k\}}(G) = \delta(G) + 1$, then for each function of any $\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u \in N[v]} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = k$ for every $u \in N[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a $\{k\}$D family on G such that $d = d_{\{k\}}(G)$, and let v be a vertex of minimum degree $\delta(G)$. Since $\sum_{u\in N[v]} f_i(u) \geq k$ for all $i \in \{1, 2, \ldots, d\}$, we obtain

$$\begin{aligned} kd &\leq \sum_{i=1}^{d} \sum_{u\in N[v]} f_i(u) \\ &= \sum_{u\in N[v]} \sum_{i=1}^{d} f_i(u) \\ &\leq \sum_{u\in N[v]} k = k(\delta(G)+1), \end{aligned}$$

and this leads to the desired bound.

If $d_{\{k\}}(G) = \delta(G) + 1$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. ☐

The next result is an immediate consequence of Theorem 11 and Observations 3 and 5.

Corollary 12. For any positive integer k and any connected graph G of order at least 2 with $\delta(G) = 1$, $d_{\{k\}}(G) = 2$.

As a further application of Theorem 11, we will prove the following Nordhaus-Gaddum type result.

Theorem 13. For every graph G of order n,

$$d_{\{k\}}(G) + d_{\{k\}}(\overline{G}) \leq n + 1.$$

If $d_{\{k\}}(G) + d_{\{k\}}(\overline{G}) = n + 1$, then G is regular.

Proof. It follows from Theorem 11 that

$$\begin{aligned} d_{\{k\}}(G) + d_{\{k\}}(\overline{G}) &\leq (\delta(G)+1) + (\delta(\overline{G})+1) \\ &= (\delta(G)+1) + (n - \Delta(G) - 1 + 1) \\ &\leq n + 1. \end{aligned}$$

If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and this inequality chain leads to the better bound $d_{\{k\}}(G) + d_{\{k\}}(\overline{G}) \leq n$, and the proof is complete. ☐

If G is isomorphic to the complete graph of order n, then $d_{\{k\}}(G) = n$ and $d_{\{k\}}(\overline{G}) = 1$, by Observation 2. Thus $d_{\{k\}}(K_n) + d_{\{k\}}(\overline{K_n}) = n + 1$. This example shows that Theorems 13 is sharp.

Corollary 14 (Cockayne and Hedetniemi [3] 1977)**.** If G is a graph of order $n \geq 1$, then $d(G) + d(\overline{G}) \leq n + 1$.

3 The $\{k\}$-domatic numbers of cycles, fans, wheels and grids

Theorem 11 and Observation 3 immediately lead to the next result.

Corollary 15. If G is a path of order $n \geq 2$, then $d_{\{k\}}(G) = 2$.

Lemma 16. Let G be a graph that is isomorphic to the cycle C_n, where $n \equiv s \pmod 3$ with $0 \leq s \leq 2$. Furthermore, let k be a positive integer and $k_1 + k_2 + \cdots + k_d = k$ such that each k_i is a non-negative integer. Suppose that $\{f_1, f_2, \ldots, f_d\}$ is a family of functions with the following properties:

(i) For $i = 1, 2, \ldots, d$: $f_i\colon V(G) \to \{0, 1, \ldots, k\}$;

(ii) For every $u \in V(G)$: $\sum_{i=1}^{d} f_i(u) \le k$;

(iii) For every $v \in V(G)$ and every $i = 1, 2, \dots, d$: $\sum_{u \in N[v]} f_i(u) = k - k_i$;

(iv) under (i)-(iii): d is maximal.

Then $d \le 4$ and the equality $d = 4$ holds if and only if $s = 0$, or $s > 0$, $k_i \ne k_j$ for $i \ne j$ and each difference $k - k_i$ is divisible by 3.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a family of functions on $V(G)$ that satisfy (i)-(iv), and let v be a vertex of G. Since $\sum_{u \in N[v]} f_i(u) = k - k_i$ for all $i \in \{1, 2, \dots, d\}$, we obtain

$$(d-1)k = \sum_{i=1}^{d}(k - k_i) = \sum_{i=1}^{d} \sum_{u \in N[v]} f_i(u) = \sum_{u \in N[v]} \sum_{i=1}^{d} f_i(u) \le \sum_{u \in N[v]} k = 3k, \tag{4}$$

and this leads to $d \le 4$.

Let $V(G)$ be given by $u_1, u_2, \dots, u_n$.

Assume first that $n \equiv 0 \pmod 3$. Define $f_j : V(G) \to \{1, 2, \dots, k\}$ by $f_j(u_i) = k$ for $i \equiv j \pmod 3$ and $f_j(u_i) = 0$ otherwise for $j \in \{1, 2, 3\}$. Furthermore, define $f_4(u_i) = 0$ for every $i = 1, 2, \dots, n$. Since $n \equiv 0 \pmod 3$, it is straightforward to verify that $\{f_1, f_2, f_3, f_4\}$ fulfills conditions (i)-(iii). Therefore $d \ge 4$ and thus, $d = 4$.

Assume second that $n \equiv s \pmod 3$ with $s > 0$, i.e. $n = 3\ell + s$ with $s \in \{1, 2\}$. If $k_i \ne k_j$ for $i \ne j$ and each difference $k - k_i$ is divisible by 3, define $f_i(u_j) = (k - k_i)/3$ for every i and j. Then it is straightforward to verify that $\{f_1, f_2, f_3, f_4\}$ fulfills conditions (i)-(iii). Therefore $d \ge 4$ and thus, $d = 4$.

Now suppose that $\{f_1, f_2, f_3, f_4\}$ is a family of functions that fulfills (i)-(iv). In addition, for $m \in \{0, 1, 2\}$ let

$$A_m = \{i : i \equiv m + 3j \pmod n, j \in \mathbb{N}_0\}$$

be the set of indices i we derive from m by adding a multiple of 3 modulo n. Let $a \ge 1$ and $b \ge 1$ be the smallest integers such that $3a = bn = b[3\ell + s]$. Then 3 divides b and $3\ell + s$ divides a and therefore $a = 3\ell + s$ and $b = 3$. It follows that $A_j = A_{j'} = \{1, 2, \dots, n\}$ for every pair $j, j' \in \{0, 1, 2\}$. Since $\sum_{x \in N[u_j]} f_i(x) = k - k_i$ for each $j \in \{1, 2, \dots, n\}$ and $i \in \{1, 2, 3, 4\}$, it follows that $f_i(u_{j_0}) = f_i(u_{j_1})$ for every pair of indices j_0, j_1 and every $i \in \{1, 2, 3, 4\}$. This implies that

$$k - k_i = \sum_{x \in N[u_2]} f_i(x) = f_i(u_1) + f_i(u_2) + f_i(u_3) = 3f_i(u_1)$$

for every $i \in \{1, 2, 3, 4\}$ and hence $f_i(u_j) = (k - k_i)/3$ for every i and every j. Therefore each difference $k - k_i$ is divisible by 3, and $k_i \ne k_j$ for $i \ne j$, since $f_i \ne f_j$ for $i \ne j$. □

With $k_1 = k_2 = k_3 = 0$ and $k_4 = k$ we derive the following corollary.

Corollary 17. Let $k \ge 1$ be an integer, and let C_n be a cycle of length n. If $n \equiv 0 \pmod 3$, then $d_{\{k\}}(C_n) = 3$, and if $n \not\equiv 0 \pmod 3$, then $d_{\{k\}}(C_n) = 2$.

The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is the disjoint union of G_1 and G_2 together with all possible edges connecting a vertex of G_1 with a vertex of G_2.

Lemma 18. If G is a graph, then $d_{\{k\}}(K_1 + G) \ge 1 + d_{\{k\}}(G)$.

Proof. If $f_1, f_2, \dots, f_d$ is a $\{k\}$D-family of G, then $\{f_1, f_2, \dots, f_d, f_{d+1}\}$ is a $\{k\}$D-family of $K_1 + G$, where $f_{d+1}(x) = \{k\}$ for $x \in V(K_1)$ and $f_{d+1}(y) = 0$ for $y \in V(G)$. □

For the classical domatic number, the equality is true.

Theorem 19 (Chang [1] 1994)**.** If G is a graph, then $d(K_1 + G) = 1 + d(G)$.

A *fan* and a *wheel* is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle, respectively. I.e., a graph of order n is a fan if it isomorphic to $K_1 + P_{n-1}$ $(n \geq 3)$, and it is a wheel if it is isomorphic to $K_1 + C_{n-1}$ $(n \geq 4)$.

Theorem 20. If G is a fan of order $n \geq 3$, then $d_{\{k\}}(G) = 3$.

Proof. Let $G = K_1 + P_{n-1}$. According to Theorem 11, $d_{\{k\}}(G) \leq \delta(G) + 1 = 3$. By Corollary 15 and Theorem 18, $d_{\{k\}}(G) \geq d_{\{k\}}(P_{n-1}) + 1 = 3$ and therefore $d_{\{k\}}(G) = 3$. □

Theorem 21. Let G be a wheel of order $n \geq 4$. If $n \equiv 1 \pmod 3$, then $d_{\{k\}}(G) = 4$. If $n \not\equiv 1 \pmod 3$, then $d_{\{k\}}(G) = 4$ if and only if $k \equiv 0 \pmod 3$ and $k \geq 18$, or $k \equiv 1 \pmod 3$ and $k \geq 22$, or $k \equiv 2 \pmod 3$ and $k \geq 26$. In all other cases $d_{\{k\}}(G) = 3$.

Proof. Let $G = K_1 + C_{n-1}$ with $V(K_1) = \{x\}$ and $V(C_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$. According to Theorem 11, $d_{\{k\}}(G) \leq \delta(G) + 1 = 4$.

Note that k can be written as a sum

$$\begin{aligned} &k = k_1 + k_2 + k_3 + k_4, \\ &k_i \in \mathbb{N}_0, \\ &k_i \neq k_j \text{ for } i \neq j, \\ &3 \text{ divides } k - k_i \text{ for } i = 1,2,3,4 \end{aligned} \tag{5}$$

if and only if $k \equiv 0 \pmod 3$ and $k \geq 18$, or $k \equiv 1 \pmod 3$ and $k \geq 22$, or $k \equiv 2 \pmod 3$ and $k \geq 26$.

Hence, if k fulfills (5) or if $n \equiv 1 \pmod 3$, then $d_{\{k\}}(G) \geq 4$ by Lemma 16.

Now suppose that $n \not\equiv 1 \pmod 3$ and k does not fulfill (5). Define $f_1(x) = k$, $f_1(u_j) = 0$ for every j, $f_2(x) = f_3(x) = 0$, $f_i(u_j) = k$ if $i \equiv j \pmod 2$ and $f_i(u_j) = 0$ if $i \not\equiv j \pmod 2$ for $i = 2, 3$ and every j. Then it is easy to see that $\{f_1, f_2, f_3\}$ is a $\{k\}$D-family of G and therefore $d_{\{k\}}(G) \geq 3$.

Assume that $d_{\{k\}}(G) = 4$ and let $\{f_1, f_2, f_3, f_4\}$ be a $\{k\}$D-family of G. Let $k_i = f_i(x)$ and $f'_i(u_j) = f_i(u_j) - k_i$ for $i = 1, 2, 3, 4$ and every j. By Theorem 11, $k = k_1 + k_2 + k_3 + k_4$ and $\{f'_1, f'_2, f'_3, f'_4\}$ fulfills conditions (i)-(iv) of Lemma 16. It follows that $k_i \neq k_j$ for $i \neq j$ and each difference $k - k_i$ is divisible by 3, a contradiction to our assumption. □

The *cartesian product* $G = G_1 \times G_2$ of two disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The cartesian product of two paths $P_r = x_1x_2 \ldots x_r$ and $P_t = y_1y_2 \ldots y_t$ is called a *grid*.

In 1994, Chang [1] determined the domatic number of grids.

Theorem 22 (Chang [1] 1994)**.** *Let $G = P_r \times P_t$ be a grid of order $n = rt$ such that $2 \leq r \leq t$. Then $d(P_2 \times P_2) = d(P_2 \times P_4) = 2$ and $d(P_r \times P_t) = 3$ in all other cases.*

The above theorem can be used to determine the $\{k\}$-domatic number of grids.

Theorem 23. *Let $G = P_r \times P_t$ be a grid of order $n = rt$ such that $2 \leq r \leq t$. Then $d_{\{k\}}(P_2 \times P_2) = 2$ and $d_{\{k\}}(P_r \times P_t) = 3$ in all other cases.*

Proof. According to Observation 1 and Theorem 11,

$$d(P_r \times P_t) \leq d_{\{k\}}(P_r \times P_t) \leq \delta(G) + 1 = 3.$$

By Theorem 22, $d(P_r \times P_t) \geq 3$ whenever $(r, t) \notin \{(2, 2), (2, 4)\}$. If $r = t = 2$, then G is a 4-cycle and $d_{\{k\}}(C_4) = 2$ by Corollary 17. It remains to consider the case $r = 2$, $t = 4$. Let $k = 3s + r$, where s is a nonnegative integer and $r \in \{0, 1, 2\}$. For $i = 1, 2$, we define $f_i((x_i, y_1)) = f_i((x_i, y_4)) = s + 1$, $f_i((x_{3-i}, y_1)) = f_i((x_{3-i}, y_4)) = s - 1 + \lceil \frac{r}{2} \rceil$, and $f_i((x_1, y_2)) = f_i((x_1, y_3)) = f_i((x_2, y_2)) = f_i((x_2, y_3)) = s + \lfloor \frac{r}{2} \rfloor$. Furthermore, let $f_3((x_i, y_j)) = k - f_1((x_i, y_j)) - f_2((x_i, y_j))$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$. Then $\{f_1, f_2, f_3\}$ is a $\{k\}$D-family of $P_r \times P_t$. □

References

[1] G.J. Chang, *The domatic number problem*, Discrete Math. **125** (1994), 115–122.

[2] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga, and J.H. van Vuuren, *Protection of a graph*, Util. Math. **67** (2005), 19–32.

[3] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977), 247–261.

[4] G.S. Domke, S.T. Hedetniemi, R.C. Laskar, and G.H. Fricke, *Relationships between integer and fractional parameters of graphs*, In Y. Alavi, G. Chartrand, O.R. Oellermann and J. Schwenk, (Eds.), Graph Theory, Combinatorics, and Applications, Proceedings of the Sixth Quadrennial International Conference on the Theory and Ap- plications of Graphs, vol. 1, pages 371–387, (Kalamazoo, MI 1988), Wiley Publications, 1991.

[5] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in graphs*, Marcel Dekker, Inc., New york, 1998.

[6] X. Hou and Y. Lu, *On the* $\{k\}$*-domination number of Cartesian products of graphs*, Discrete Math. **309** (2009), 3413–3419.

[7] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.

[8] B. Zelinka, *On* k*-domatic numbers of graphs*, Czechoslovak Math. J. **33** (1983) no. 2, 309–313.