Cycles avoiding a Color in Colorful Graphs

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Abstract

The Ramsey numbers of cycles imply that every 2-edge-colored complete graph on n vertices contains monochromatic cycles of all lengths between 4 and at least $\frac{2}{3}n$. We generalize this result to $k \geq 3$ colors by showing that every k-edge-colored complete graph on $n \geq 6$ vertices contains $(k-1)$-edge-colored cycles of all lengths between 3 and at least $\left(\frac{2k-2}{2k-1} - \frac{2k-4}{\sqrt{n}}\right)n$.

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1 Introduction

We consider finite, simple, and undirected graphs and use standard terminology as in [4].

The Ramsey number $R(C_\ell)$ of the cycle C_ℓ of length ℓ is the minimum integer n such that every 2-edge-colored complete graph of order n contains a monochromatic C_ℓ, that is, a cycle of length l whose edges are all of the same color. In a combined effort of several researchers [2, 3, 8, 11], see also [9], the Ramsey number $R(C_\ell)$ was determined precisely:

$$R(C_\ell) = \begin{cases} 6 & , \ell \in \{3, 4\}, \\ 2\ell - 1 & , \ell \geq 5 \text{ and } \ell \text{ is odd}, \\ \frac{3\ell}{2} - 1 & , \ell \geq 6 \text{ and } \ell \text{ is even}. \end{cases} \tag{1}$$

As shown by Faudree et al. [7], (1) implies that every 2-edge-colored complete graph on n vertices with $n \geq 6$ contains a monochromatic cycle of length at least $\left\lceil \frac{2n}{3} \right\rceil$.

There are several reasonable ways to generalize this result to more than two colors. First, one can consider monochromatic cycles in k-edge-colored complete graphs. The corresponding Ramsey numbers for 3 colors [1, 10] for instance imply that every 3-edge-colored complete graph on n vertices contains a monochromatic cycle of length at least $\frac{3n}{8} + O(1)$. Second, one can consider cycles in k-edge-colored complete graphs that avoid one of the k colors, that is, $(k-1)$-edge-colored cycles in k-edge-colored complete graphs. In the present paper we present contributions to this second line of research.

Note that a k'-edge-colored cycle is not supposed to contain, for each of k' distinct colors, an edge colored with that color. Instead, a cycle, or more generally a graph, is k'-edge-colored if at most k' distinct colors occur on its edges. Furthermore, none of the edge-colorings we consider needs to be proper.

In Section 2 we state and discuss our main results and in Section 3 we provide their proofs.

2 Results

By a classical extremal result of Erdős and Gallai [6], every graph of order n that contains no cycle of length at least c has most $\frac{1}{2}(c-1)(n-1)$ edges. Since every k-edge-colored complete graph on n vertices contains a $(k-1)$-edge-colored spanning subgraph with at least $\frac{k-1}{k}\binom{n}{2}$ edges, Erdős and Gallai's result implies the existence of a $(k-1)$-edge-colored cycle of length at least $\frac{k-1}{k}n$. The following theorem is our main result. It improves the factor $\frac{k-1}{k}$, whose proof purely relied on a density argument.

Theorem 1. *Let k and n be positive integers. Every k-edge-colored complete graph on n vertices contains a $(k-1)$-edge-colored cycle of length at least*

$$\left(\frac{2k-2}{2k-1}-\frac{2k-4}{\sqrt{n}}\right)n.$$

We fear that the factor $\frac{2k-2}{2k-1}$ is still far from best possible.

In fact, let k and n_0 be positive integers and let G be a complete graph whose vertex set is the disjoint union of k sets $V_1,\ldots,V_k$ with $|V_i|=n_0 2^{i-1}$ such that each edge uv of G with $u\in V_i$ and $v\in V_j$ is of color $\min\{i,j\}$. For $i\in[k]=\{1,\ldots,k\}$, removing all edges of color i from G, results in a graph in which V_i is an independent set whose neighborhood is contained in $V_1\cup\ldots\cup V_{i-1}$. Since $|V_i|-|V_1\cup\ldots\cup V_{i-1}|=n_0$, every cycle in G that avoids edges of color i, misses at least n_0 vertices. Therefore, the k-edge-colored complete graph G on $n=n_0(2^0+2^1+\cdots+2^{k-1})=n_0(2^k-1)$ vertices contains no $(k-1)$-edge-colored cycle of length more than $\frac{2^k-2}{2^k-1}n$.

We suspect that this construction is best possible.

Our second result implies the existence of $(k-1)$-edge-colored cycles of all lengths between 3 and the maximum length.

Theorem 2. *Let k, n, and ℓ be integers with $k\geq 3$, $n\geq 6$, and $\ell\geq 4$. If a k-edge-colored complete graph on n vertices contains a $(k-1)$-edge-colored cycle of length ℓ, then it contains a $(k-1)$-edge-colored cycle of length $\ell-1$.*

The rest of this paper is devoted to the proofs of Theorems 1 and 2.

3 Proofs

We begin by observing a consequence of the following result of Károlyi and Rosta [9].

Lemma 3 (Károlyi and Rosta [9])**.** *Let k and m be positive integers such that k is even and $m>4$. If we color the edges of a complete graph on $m+\frac{1}{2}k-1$ vertices red and blue, then there exists either a monochromatic cycle of length at least m or a blue cycle of length k.*

We exploit the slight asymmetry of the last result.

Corollary 4. *Let G be a complete graph on n vertices whose edges are colored red and blue and let $\varepsilon\in\mathbb{R}$. If G does not have a monochromatic cycle of length at least $\frac{2}{3}n+\varepsilon$, then G has a blue cycle of length at least $\frac{2}{3}n-2\varepsilon$.*

Proof. If $\varepsilon \leq 0$, then the mentioned result from [7] implies the existence of a monochromatic cycle of length at least $\frac{2}{3}n + \varepsilon$. Hence we may assume that $\varepsilon > 0$. Let k be the smallest even integer greater or equal than $\frac{2}{3}n - 2\varepsilon$ and let $m = n - \frac{1}{2}k + 1$. Then $n = m + \frac{1}{2}k - 1$. By Lemma 3, G contains a monochromatic cycle of length at least m or a blue cycle of length k. Since $m > n - \frac{1}{2}\left(\frac{2}{3}n - 2\varepsilon + 2\right) + 1 = \frac{2}{3}n + \varepsilon$, the former is not possible. $\square$

The next lemma contains the technical core of our argument.

Lemma 5. *Let G be a complete graph whose edges are colored green, red, and blue and let $p \in \mathbb{R}_{\geq 0}$. If C_1 is a green cycle of length n_1 and C_2 a red cycle in $G - V(C_1)$ of length n_2, then G has*

- *a green-red cycle of length at least $n_1 + n_2 - 2p$ or*
- *a green-blue cycle of length greater than*

$$n_1 + p \min\left\{\frac{1}{p+2}n_1, \frac{1}{p+1}n_2\right\} - 2$$

and a red-blue cycle of length greater than

$$n_2 + p \min\left\{\frac{1}{p+1}n_1, \frac{1}{p+2}n_2\right\} - 2$$

using only vertices of C_1 and C_2.

Proof. Let $C_1 = x_1x_2\ldots x_{n_1}x_1$ and $C_2 = y_1y_2\ldots y_{n_2}y_1$. We may assume that all green-red cycles of $G[V(C_1) \cup V(C_2)]$ are of length less than $n_1 + n_2 - 2p$. We shall show that G has a green-blue cycle of length greater than $n_1 + p\min\left\{\frac{1}{p+2}n_1, \frac{1}{p+1}n_2\right\} - 2$. A long red-blue cycle may be obtained by switching the roles of C_1 and C_2.

Suppose that all edges between C_1 and C_2 are blue. If $n_1 \geq n_2$, then

$$x_1y_1x_2y_2x_3\ldots x_{n_2}y_{n_2}x_{n_2+1}x_{n_2+2}\ldots x_1$$

is a green-blue cycle of length $n_1 + n_2$ in G, and if $n_1 < n_2$, then

$$x_1y_1x_2y_2x_3\ldots x_{n_1}y_{n_1}x_1$$

is a green-blue cycle of length $2n_1$ in G and the claim holds.

Otherwise, there is at least one edge between C_1 and C_2 that is not blue, say x_1y_1. Starting with (i_1, j_1), where $i_1 = j_1 = 1$, we construct a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$ of pairs of indices such that

(1) $1 = i_1 < i_2 < \ldots < i_q \leq n_1$ and $1 = j_1 < j_2 < \ldots < j_q \leq n_2$,

(2) $n_1 + n_2 - i_q - j_q \geq 2p$,

(3) for every $1 \leq k \leq q$, at least one of $x_{i_k-1}y_{j_k}$, $x_{i_k}y_{j_k}$ is green or red,

(4) for every $1 \leq k \leq q-1$, $j_{k+1} - j_k + 1 = i_{k+1} - i_k \geq p + 2$,

(5) for every $1 \le k \le q-1$, the path $x_{i_k+\ell}y_{j_k+\ell}x_{i_k+\ell+1}$ is blue for $1 \le \ell \le i_{k+1} - i_k - 2$, and

(6) for every $1 \le \ell \le \min\{n_1 - i_q - 1, n_2 - j_q\}$, the path $x_{i_q+\ell}y_{j_q+\ell}x_{i_q+\ell+1}$ is blue.

Note that the sequence (i_1, j_1) satisfies (1)-(5). For a positive integer s, assume that we have constructed $(i_1, j_1), (i_2, j_2), \dots, (i_s, j_s)$ satisfying (1)-(5). If $x_{i_s+\ell}y_{j_s+\ell}x_{i_s+\ell+1}$ is blue for every $1 \le \ell \le \min\{n_1 - i_s - 1, n_2 - j_s\}$, then (6) is satisfied by $(i_1, j_1), (i_2, j_2), \dots, (i_s, j_s)$ and we stop the construction. Otherwise, let $m = \min\{\ell \ge 1 \colon x_{i_s+\ell}y_{j_s+\ell}x_{i_s+\ell+1}$ is not blue$\}$. Let $j_{s+1} = m$ and $i_{s+1} = m + 1$. We shall show that $(i_1, j_1), (i_2, j_2), \dots, (i_{s+1}, j_{s+1})$ satisfy conditions (1)-(5). Note that the sequence fulfills conditions (1), (3), and (5) by construction. Furthermore, (i_{s+1}, j_{s+1}) is defined such that $i_{s+1} - i_s = j_{s+1} - j_s + 1$. Moreover, if there exist two non-blue edges xy and $x'y'$ between C_1 and C_2 such that $x, x' \in V(C_1)$ with $x \ne x'$, $y, y' \in V(C_2)$ with $y \ne y'$, and $\mathrm{dist}_{C_1}(x, x') + \mathrm{dist}_{C_2}(y, y') \le 2p + 2$, then it is easy to see that G has a green-red cycle of length at least $n_1 + n_2 - 2p$. Hence, the sequence satisfies (2) and (4).

For $1 \le k \le q-1$, let P_k be the blue path defined by the edges in (5), and P_q the blue path defined by the edges in (6). Moreover, for $1 \le k \le q$, let P'_k be the green path between the last vertex of P_k and the first vertex of P_{k+1} using the edges of C_1, where $P_{q+1} = P_1$. Then $C^* = P_1P'_1P_2P'_2P_3 \dots P_qP'_q$ is a green-blue cycle in G. If $n_1 - i_q \ge n_2 - j_q + 1$, then $q \le 1 + \frac{1}{p+1}(n_2 - 1)$ implies

$$|V(C^*)| \ge n_1 + n_2 - q > n_1 + \frac{p}{p+1}n_2 - 1.$$

Note that $j_q = i_q - q + 1$. If $n_1 - i_q \le n_2 - j_q$, then $q \le 1 + \frac{1}{p+2}(n_1 - 1)$ implies

$$\begin{aligned}|V(C^*)| &\ge n_1 + n_2 - q - ((n_2 - j_q) - (n_1 - i_q - 1))\\ &= 2n_1 - i_q + j_q - q - 1\\ &= 2n_1 - 2q\\ &> n_1 + \frac{p}{p+2}n_1 - 2.\end{aligned}$$

This completes the proof of the lemma. □

Corollary 6. *Let G be a complete graph on n vertices whose edges are colored green, red, and blue. If C_1 is a green cycle of length n_1 and C_2 a red cycle in $G - V(C_1)$ of length n_2, then G has a 2-edge-colored cycle of length at least $n_1 + n_2 - 2\sqrt{n}$.*

Proof. Let $p = \sqrt{n}$. By Lemma 5, G has

(1) a green-red cycle of length at least $n_1 + n_2 - 2\sqrt{n}$ or

(2) a green-blue cycle of length greater than $n_1 + \sqrt{n}\min\left\{\frac{1}{\sqrt{n}+2}n_1, \frac{1}{\sqrt{n}+1}n_2\right\} - 2$ and a red-blue cycle of length greater than $n_2 + \sqrt{n}\min\left\{\frac{1}{\sqrt{n}+1}n_1, \frac{1}{\sqrt{n}+2}n_2\right\} - 2$.

In case of (1), there is nothing to show. In case of (2), we assume, without loss of generality, that $n_1 \geq n_2$ and thus, $n_2 \leq \frac{1}{2}n$. Since

$$\begin{aligned} n_1 + \sqrt{n}\min\left\{\frac{1}{\sqrt{n}+2}n_1, \frac{1}{\sqrt{n}+1}n_2\right\} - 2 &\geq n_1 + \sqrt{n}\min\left\{\frac{1}{\sqrt{n}+2}n_1, \frac{1}{\sqrt{n}+2}n_2\right\} - 2 \\ &= n_1 + n_2 - \frac{2}{\sqrt{n}+2}n_2 - 2 \\ &\geq n_1 + n_2 - \frac{n}{\sqrt{n}+2} - 2 \\ &\geq n_1 + n_2 - 2\sqrt{n} \end{aligned}$$

the desired result follows. □

Lemma 7. *Let G be a complete graph whose edges are colored with $1, 2, \ldots, k$. If vw is an edge of color 1 and P is a $(k-1)$-edge-colored path of length $\ell - 1$ between v and w where $\ell \geq 3$, then G has a $(k-1)$-edge-colored cycle of length at least ℓ or $d_1(v) + d_1(w) \geq n - 1$, where $d_1(u)$ denotes the number of edges of color 1 incident with a vertex u of G.*

Proof. Let $P = u_1u_2 \ldots u_\ell$ with $u_1 = v$ and $u_\ell = w$.

If P contains an edge of color 1, then $u_1u_2 \ldots u_\ell u_1$ is a $(k-1)$-edge-colored cycle of length ℓ. Hence, we may assume that no edge of P is colored with 1.

If there exists a vertex $x \notin V(P)$ such that neither vx nor xw is colored with 1, then $u_1u_2 \ldots u_\ell x u_1$ is a $(k-1)$-edge-colored cycle of length $\ell + 1$ in G.

Furthermore, if there exists an index $i \in [\ell - 1]$ such that u_1u_{i+1} and $u_\ell u_i$ are not colored with 1, then $u_1u_{i+1}u_{i+2} \ldots u_\ell u_i u_{i-1} \ldots u_1$ is a $(k-1)$-edge-colored cycle of length ℓ in G. Hence, we may assume that for every vertex $x \notin V(P)$, at least one of vx and xw is colored with 1, and for every index $i \in [\ell - 1]$, at least one of vu_{i+1} and wu_i is colored with 1 and thus, the lemma holds. □

After these preparations we proceed to the proof of our main result.

Proof of Theorem 1. The proof is by induction on k. For $k = 1$ the statement is void and for $k = 2$ it is the main result of [7].

Let $k \geq 3$ and $c(n,k) = \left(\frac{2k-2}{2k-1} - \frac{2k-4}{\sqrt{n}}\right)n$. We assume that G does not contain a $(k-1)$-edge-colored cycle of length at least $c(n,k)$ and that G contains as few edges of color 1 as possible. Note that the number of edges of color 1 is at most $\frac{1}{k}\binom{n}{2} = \frac{1}{2k}n(n-1)$.

Claim 1. *If vw is an edge of color 1, then $d_1(v) + d_1(w) \geq n - 1$.*

Proof. By the choice of G, recoloring vw with a color different from 1 creates a $(k-1)$-edge-colored cycle of length at least $c(n,k)$ containing the edge vw. Hence $G - vw$ contains a $(k-1)$-edge-colored path P of length at least $c(n,k) - 1$ between v and w and the claim follows from Lemma 7. □

Claim 2. *If H is a subgraph of G containing only edges of color 1 such that all components of H are paths of length 1 or cycles, then H has at most $\frac{2}{k}n$ vertices.*

Proof. For $d \in \{1, 2\}$, let H_d be the subgraph of H that is induced by the vertices of degree d in H. If $|V(H)| > \frac{2}{k}n$, then, by Claim 1,

$$4|E(H)| = 2 \sum_{vw \in E(H_1)} (d_1(v) + d_1(w)) + \sum_{vw \in E(H_2)} (d_1(v) + d_1(w)) > \frac{2}{k}n(n-1),$$

that is, G has more than $\frac{1}{2k}n(n-1)$ edges of color 1, which is a contradiction. □

Let $C = v_1 v_2 \dots v_t v_1$ be a longest cycle of color 1 in G. By choosing $\varepsilon = c(n,k) - \frac{2}{3}n$ in Corollary 4, we may assume that $t \geq 2n - 2c(n,k)$.

Claim 3. *There is no edge of color* 1 *in* $G - V(C)$.

Proof. Suppose that there exist edges of color 1 in $G - V(C)$.

Let H be a subgraph of $G - V(C)$ containing only edges of color 1 such that

(1) every component of H is a path of length 1 or a cycle;

(2) under condition (1), the number of vertices of H is maximal;

(3) under conditions (1) and (2), the number of components of H is minimal.

By assumption, H is not empty. Note that $|V(C)| + |V(H)| \leq \frac{2}{k}n$ by Claim 2. For $d \in \{1, 2\}$, let H_d be the subgraph of H that is induced by its vertices of degree d.

We shall show that there is a cycle in H. Assume, for a contradiction, that $H = H_1$. Let uw be an edge in H. If there exists a vertex x in $G - (V(C) \cup V(H))$ such that ux and wx are of color 1, then $uxwu$ is a cycle of color 1, a contradiction to condition (2). Hence, for every vertex x in $G - (V(C) \cup V(H))$, there is at most one edge of color 1 leading from $\{u, w\}$ to x. If $xy \neq uw$ is an edge in H_1 such that ux and wy are of color 1, then $uxywu$ is a cycle of color 1, a contradiction to condition (3). Thus, for every component $xy \neq uw$ in H_1, there are at most two edges of color 1 between $\{u, w\}$ and $\{x, y\}$. If there exists an index $1 \leq i \leq t$ and a vertex x in $G - V(C)$ such that xv_i and xv_{i+1} are of color 1, then C is not longest, a contradiction. Furthermore, if there exists an index $1 \leq i \leq t$ such that uv_i and one of wv_{i+1}, wv_{i+2} are of color 1, then C is not longest, again a contradiction. Hence, for every index i, there are at most two edges of color 1 between $\{v_i, v_{i+1}, v_{i+2}\}$ and $\{u, w\}$ and thus, the number of edges of color 1 between $\{u, w\}$ and C is at most $\frac{2}{3}|V(C)|$. It follows that

$$d_1(u) + d_1(w) \leq \frac{2}{3}|V(C)| + |V(H_1)| + n - |V(C)| - |V(H_1)| = n - \frac{1}{3}|V(C)| < n - 1,$$

a contradiction to Claim 1. So, there exists a cycle in H.

Let $C' = w_1 w_2 \dots w_r w_1$ be a cycle in H. Note that w_1 and w_r are the endvertices of the path $C' - w_1 w_r$ of length $|V(C')| - 1 \geq 2$.

If there exists an index $1 \leq i \leq t$ such that $w_r v_i$ and one of $w_1 v_{i+1}, w_1 v_{i+2}, w_1 v_{i+3}$ are of color 1, then C is not longest, a contradiction. Hence, for every index i, there are at most two edges of color 1 between $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ and $\{w_1, w_r\}$ and thus, the number of edges of color 1 between $\{w_1, w_r\}$ and C is at most $\frac{1}{2}|V(C)|$.

If there is no edge of color 1 between $\{w_1, w_r\}$ and $G - (V(C) \cup V(H))$, then

$$\begin{aligned} d_1(w_1) + d_1(w_r) &\leq \frac{1}{2}|V(C)| + 2|V(H)| - 2 \\ &= 2(|V(C)| + |V(H)|) - \frac{3}{2}|V(C)| - 2 \\ &\leq \frac{4}{k}n - \frac{3}{2}(2n - 2c(n,k)) - 2 \\ &= n\left(\frac{4}{k} + 3\left(\frac{c(n,k)}{n} - 1\right)\right) - 2. \end{aligned}$$

Note that $3\left(\frac{c(n,k)}{n} - 1\right) < 0$. If $k \geq 4$, then $\frac{4}{k} + 3\left(\frac{c(n,k)}{n} - 1\right) < 1$ and if $k = 3$, then $\frac{4}{k} + 3\left(\frac{c(n,k)}{n} - 1\right) < \frac{11}{15}$. In both cases, it follows that $d_1(w_1) + d_1(w_r) < n - 2$, a contradiction to Claim 1. So, there is at least one edge of color 1 between $\{w_1, w_r\}$ and $G - (V(C) \cup V(H))$.

Without loss of generality, let $w_r x$ be an edge of color 1, where $x \in V(G) - (V(C) \cup V(H))$. If r is odd, then replacing C' by $w_1w_2, w_3w_4, \ldots, w_{r-2}w_{r-1}, w_r x$ yields a contradiction to condition (2). Hence, r is even. If there is an edge $w_1 y$ of color 1 such that $y \neq x$ and $y \in V(G) - (V(C) \cup V(H))$, then replacing C' by $w_2w_3, w_4w_5, \ldots, w_{r-2}w_{r-1}, w_1 y, w_r x$ again yields a contradiction to condition (2). It follows that there is no edge of color 1 between w_1 and $G - (V(C) \cup V(H))$. If there is an index i such that $w_1 w_{2i-1}$ is of color 1, then replacing C' by $w_1 w_2 \ldots w_{2i-1} w_1$ and $w_{2i}w_{2i+1}, w_{2i+2}w_{2i+3}, \ldots, w_{r-2}w_{r-1}, w_r x$ yields another contradiction to condition (2). Therefore there is no edge of color 1 between w_1 and $\{w_3, w_5, \ldots, w_{r-1}\}$. It follows that there are at most $|V(C')| - 1$ edges of color 1 between w_r and $C' - w_r$ and at most $\frac{1}{2}|V(C')|$ edges of color 1 between w_1 and $C' - w_1$.

If $C'' = u_1 u_2 \ldots u_s u_1$ is another cycle in H and there exists an index $1 \leq i \leq s$ such that $w_1 u_i$ and $w_r u_{i+1}$ are of color 1, then $w_r u_{i+1} u_{i+2} \ldots u_i w_1 w_2 \ldots w_r$ is a cycle of color 1 with vertex set $V(C') \cup V(C'')$, a contradiction to condition (3). Hence, for every index i, there are at most two edges of color 1 between $\{u_i, u_{i+1}\}$ and $\{w_1, w_r\}$ and thus, the number of edges of color 1 between $\{w_1, w_r\}$ and $H_2 - V(C')$ is at most $|V(H_2)| - |V(C')|$.

If xy is a component of H_1 and $w_1 x$ and $w_r y$ are of color 1, then $w_r y x w_1 w_2 \ldots w_r$ is a cycle of color 1 with vertex set $V(C') \cup \{x, y\}$, a contradiction to condition (3). Hence, for every component xy of H_1, there are at most two edges of color 1 between $\{x, y\}$ and $\{w_1, w_r\}$ and thus, the number of edges of color 1 between $\{w_1, w_r\}$ and H_1 is at most $|V(H_1)|$.

Finally, note that due to the maximality of H, there is no vertex x in $G - (V(C) \cup V(H))$ such that $w_1 x$ and $w_r x$ are of color 1. It follows that there are at most $n - |V(C)| - |V(H)|$ edges of color 1 between $\{w_1, w_r\}$ and $G - (V(C) \cup V(H))$.

Combining all these observations we conclude that

$$\begin{aligned} n - 1 &\leq d_1(w_1) + d_1(w_r) \\ &\leq \tfrac{1}{2}|V(C)| + \left(\tfrac{3}{2}|V(C')| - 1\right) + (|V(H_2)| - |V(C')|) + |V(H_1)| + (n - |V(C)| - |V(H)|) \\ &= n + \tfrac{1}{2}|V(C')| - \tfrac{1}{2}|V(C)| - 1, \end{aligned}$$

and thus,

$$|V(C')| \geq |V(C)| \geq 2n - 2c(n,k) > \frac{2}{2k-1}n.$$

This bound yields the contradiction

$$\frac{4}{2k-1}n \leq |V(C)| + |V(C')| \leq |V(C)| + |V(H)| \leq \frac{2}{k}n$$

and the proof is complete. $\square$

By Claim 3, the subgraph $G - V(C)$ contains no edges of color 1. Hence, by the induction hypothesis, $G - V(C)$ contains a $(k-2)$-edge-colored cycle $\overline{C} = w_1 w_2 \dots w_s w_1$ of length $s \geq c(n-t, k-1)$.

Claim 4. *G has a $(k-1)$-edge-colored cycle of length at least $c(n,k)$.*

Proof. By Corollary 6, G has a $(k-1)$-edge-colored cycle of length at least

$$\begin{aligned} t + s - 2\sqrt{n} &\geq t + c(n-t, k-1) - 2\sqrt{n} \\ &= t + \frac{2k-4}{2k-3}(n-t) - (2k-6)\sqrt{n-t} - 2\sqrt{n} \\ &\geq \frac{2k-4}{2k-3}n + \left(1 - \frac{2k-4}{2k-3}\right) t - (2k-4)\sqrt{n} \\ &\geq \frac{2k-4}{2k-3}n + \frac{1}{2k-3}(2n - 2c(n,k)) - (2k-4)\sqrt{n} \\ &\geq \frac{2k-4}{2k-3}n + \left(\frac{1}{2k-3}\right)\left(\frac{2}{2k-1}\right) n - (2k-4)\sqrt{n} \\ &= c(n,k). \end{aligned}$$

$\square$

Claim 4 completes the proof of the theorem. $\square$

We conclude this section with the proof of our second theorem.

Proof of Theorem 2. Let G be a k-edge-colored complete graph on n vertices and let $C = v_1 \dots v_\ell v_1$ be a $(k-1)$-edge-colored cycle of length ℓ in G. We denote the edge-colors by $1, \dots, k$ and assume that C has no edge of color k. Since $n \geq 6 = R(C_3)$, the statement is trivial for $\ell = 4$. Hence, we may assume that $\ell \geq 5$.

If some edge of the form $v_i v_{i+2}$ is not colored k, then $v_1 \dots v_i v_{i+2} \dots v_\ell v_1$ is a $(k-1)$-edge-colored cycle of length $\ell - 1$ in G. Hence we may assume that all edges of the form $v_i v_{i+2}$ are colored k. If ℓ is odd, say $\ell = 2r + 1$, then $v_1 v_5 v_7 \dots v_{2r+1} v_2 v_4 v_6 \dots v_{2r} v_1$ is a 2-edge-colored cycle of length $\ell - 1$ in G. (This observation corresponds essentially to Lemma 2.1 in [9].) Hence we may assume that ℓ is even, say $\ell = 2r$.

All edges of the two disjoint cycles $C_1 = v_1 v_3 \dots v_{2r-1} v_1$ and $C_2 = v_2 v_4 \dots v_{2r} v_2$ are colored k. If $k \geq 4$ or some edge between $V(C_1)$ and $V(C_2)$, say $v_1 v_2$, is colored k, then $(C_1 - v_3) \cup (C_2 - v_2 v_4) + v_1 v_2 + v_4 v_5$ is a $(k-1)$-edge-colored cycle of length $\ell - 1$ in G. Hence, we may assume that $k = 3$ and that no edge between $V(C_1)$ and $V(C_2)$ is colored k. If some edge of the form $v_i v_{i+2s}$ is not colored k, say for $i = 1$, then $v_1 v_2 \dots v_{2s-1} v_\ell v_{\ell-1} \dots v_{2s+1} v_1$ is a $(k-1)$-edge-colored cycle of length $\ell - 1$ in G. Hence $V(C_1)$ and $V(C_2)$ induce complete subgraphs G_1 and G_2 of G with all edges colored k. Since $k = 3$ and $\ell \geq 6$, there are two independent edges of the same color between $V(C_1)$ and $V(C_2)$. These two edges together with suitable paths in G_1 and G_2 form a 2-edge-colored cycle of length $\ell - 1$ in G. $\square$

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