k-restricted edge-connectivity in triangle-free graphs

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Abstract

Let $G = (V, E)$ be a λ_k-connected graph. G is called λ_k-optimal, if its k-restricted edge-connectivity $\lambda_k(G)$ equals its minimum k-edge degree. G is called *super*-λ_k if every λ_k-cut isolates a connected subgraph of order k.

Firstly, we give a lower bound on the order of 2-fragments in triangle-free graphs that are *not* λ_2-optimal. Secondly, we present an Ore-type condition for triangle-free graphs to be λ_3-optimal. Thirdly, we prove a lower bound on the order of k-fragments in triangle-free λ_k-connected graphs, and use it to show that triangle-free graphs with high minimum degree are λ_k-optimal and *super*-λ_k.

Keywords: triangle-free, k-restricted edge-connectivity, λ_k-optimal, super-λ_k.

1 Terminology and introduction

We consider finite graphs without loops and multiple edges. For any graph G the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. We define the *order* of G by $n = n(G) = |V(G)|$ and the *size* by $m = m(G) = |E(G)|$.

If G is a graph, then the *degree* $d(v) = d_G(v)$ of a vertex v is the number of vertices of G adjacent with v. Therefore, $\delta = \delta(G) = \min\{d(v) \colon v \in V(G)\}$ denotes the *minimum degree* of G. We call the vertex set $N_G(v) = N(v)$ of all neighbors of a vertex $v \in V(G)$ the *open neighborhood* and $N_G[v] = N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. If $A \subset V(G)$, then $N(A) = \bigcup_{v \in A} N(v) \setminus A$, $N[A] = \bigcup_{v \in A} N[v]$, and $G[A]$ is the graph induced by A. Furthermore, we write $G - H = G[V(G) \setminus V(H)]$ for a subgraph H of G, and for a subset $S \subset E(G)$ of edges, $G - S$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus S$.

A graph is called *triangle-free* if it contains no cycle of length three. A vertex set $S \subset V(G)$ is called *independent (vertex) set* if its induced subgraph contains no edges. We call a graph *bipartite* if its vertex set can be partitioned into two independent sets. A vertex set of cardinality k is called a *k-clique* if its induced subgraph is isomorphic to the complete graph on k vertices.

An *edge-cut* in a connected graph G is a set S of edges of G such that $G - S$ is disconnected. An edge-cut S is called a *k-restricted edge-cut* if every component of $G - S$ has at least k vertices. Assuming that G has k-restricted edge-cuts, the *k-restricted edge-connectivity* of G, denoted by $\lambda_k(G)$, is defined as the minimum cardinality over all k-restricted edge-cuts of G, i.e.

$$\lambda_k(G) = \min\{|S| \colon S \subset E(G) \text{ is a k-restricted edge-cut}\}.$$

A connected graph G is called *λ_k-connected* if λ_k exists. If $[X, \overline{X}]$ denotes the edges between a vertex set $X \subset V(G)$ and its complement $\overline{X} = V(G) \setminus X$, then a k-restricted edge-cut $[X, \overline{X}]$ is called λ_k-cut if $|[X, \overline{X}]| = \lambda_k(G)$. It is clear that for any λ_k-cut $[X, \overline{X}]$, the graph $G - [X, \overline{X}]$ has exactly two connected components.

Let $[X, \overline{X}]$ be a λ_k-cut, then X is called a *k-fragment* of G. Let

$$r_k(G) = \min\{|X| : X \text{ is a k-fragment of } G\}.$$

Obviously, $k \leq r_k(G) \leq |V(G)|/2$. A k-fragment X is called a *k-atom* of G if $|X| = r_k(G)$.

For every positive integer k, the *minimum k-edge-degree* is defined as

$$\xi_k(G) = \min\{|[X, \overline{X}]| \colon |X| = k \text{ and } G[X] \text{ is connected}\}.$$

A λ_k-connected graph G with $\lambda_k(G) \leq \xi_k(G)$ is said to be *optimally k-restricted edge-connected* (for short *λ_k-optimal*) if $\lambda_k(G) = \xi_k(G)$. λ_1 and λ_2 correspond to the *edge-connectivity* and *restricted edge-connectivity*, respectively, and accordingly ξ_1 and ξ_2 are also known as the *vertex degree* and the *edge degree*. Together with a result from Bonsma, Ueffing and Volkmann [1] on λ_3 we have $\lambda_k(G) \leq \xi_k(G)$ for $1 \leq k \leq 3$ and all graphs G aside from a class of exceptions for $k = 3$ determined in [1]. Also in [1] the authors give a number of examples, which show that $\lambda_k(G) \leq \xi_k(G)$ is not true in general for $k \geq 4$.

A graph G is called *super-λ_k* if every λ_k-cut isolates a connected subgraph of order k. By definition, if G is super-λ_k, then G is λ_k-optimal. However, the converse is not true. For example, a cycle of length $n \geq 2k + 2$ is λ_k-optimal but not super-λ_k.

The *restricted edge-connectivity* was first introduced and studied by Esfahanian and Hakimi [2] in 1988. It is a special case of a quite general concept of *conditional edge-connectivity*, proposed by Harary [4] in 1983 as a measurement for fault tolerance of interconnection networks. The k-restricted edge-connectivity we consider in this paper is due to Fàbrega and Fiol [3].

In the following sections we present recent as well as new results concerning the k-restricted edge-connectivity and λ_k-optimality in triangle-free graphs. Therefore, Section 2 will deal with λ_2-optimality, where we present a lower bound on the 2-fragments of triangle-free graphs that are not λ_2-optimal in terms of ξ_2. In Section 3 we give a sufficient condition for triangle-free graphs, which fulfill a degree condition for non-adjacent vertices, to be λ_3-optimal. Finally, in Section 4 we consider the restricted k-edge-connectivity of triangle-free graphs.

2 λ_2-optimality in triangle-free graphs

Recently, Yuan and Liu [8] gave the following sufficient condition for triangle-free graphs to be λ_2-optimal.

Theorem 2.1 (Yuan & Liu [8] 2010)**.** *Let G be a connected triangle-free graph of order $n \geq 4$. If $d(u) + d(v) \geq 2\left\lfloor \frac{n+2}{4} \right\rfloor + 1$ for each pair u, v of vertices of distance two, then G is λ_2-optimal.*

In the same work the authors provided a sufficient criterion for a λ_k-connected graph G with $\lambda_k(G) \leq \xi_k(G)$ to be λ_k-optimal.

Lemma 2.2 (Yuan & Liu [8] 2010)**.** *Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$, and let U be a λ_k-fragment of G. If there is a connected subgraph H of order k in $G[U]$ such that*

$$|[V(H), U \setminus V(H)]| \leq |[U \setminus V(H), \overline{U}]|,$$

then G is λ_k-optimal.

If a λ_2-connected graph is not λ_2-optimal, we have the following tight lower bound on the cardinality of the 2-fragments of its 2-edge-cuts.

Theorem 2.3. *Let G be a λ_2-connected and triangle-free graph with minimum degree $\delta \geq 2$. If G is not λ_2-optimal, then $r_2(G) \geq \max\{3, \frac{1}{\delta}((\delta-1)\xi_2(G) + 2\delta + 1)\}$.*

Proof. Note that $\lambda_2(G) \leq \xi_2(G)$. Let U be a λ_2-atom of G. If $|U| = 2$, then $\xi_2(G) \leq |[U, \overline{U}]| = \lambda_2(G)$, which is a contradiction. So assume that $r_2(G) = |U| \geq 3$. For $\delta = 1$ it is $r_2(G) \geq 3 = \frac{1}{\delta}((\delta-1)\xi_2(G) + 2\delta + 1)$. Therefore, we assume $\delta \geq 2$.

Let xy be an edge in U such that $|[\{x, y\}, \overline{U}]|$ is minimal among all edges in U. Let $X = (N(x) \cap U) \setminus \{y\}$ and $Y = (N(y) \cap U) \setminus \{x\}$. Then $X \cup Y \neq \emptyset$ and, since G is triangle-free, we have $X \cap Y = \emptyset$. The choice of xy implies that $|[v, \overline{U}]| \geq |[y, \overline{U}]|$ ($\lozenge$) and $|[w, \overline{U}]| \geq |[x, \overline{U}]|$ for all vertices $v \in X$ and $w \in Y$.

If $|[x,\overline{U}]| \geq 1$ and $|[y,\overline{U}]| \geq 1$, then

$$|[\{x,y\}, U \setminus \{x,y\}]| = |X| + |Y| \leq |[X \cup Y, \overline{U}]| \leq |[U \setminus \{x,y\}, \overline{U}]|$$

and thus, G is λ_2-optimal by Lemma 2.2, a contradiction. So assume without loss of generality that $|[x,\overline{U}]| = 0$.
Note that

$$\begin{aligned}\xi_2(G) &\leq |[\{x,y\}, V(G) \setminus \{x,y\}]| \\ &= |[\{x,y\}, U \setminus \{x,y\}]| + |[\{x,y\}, \overline{U}]| \leq r_2(G) - 2 + |[\{x,y\}, \overline{U}]|\end{aligned}$$

and thus,

$$r_2(G) \geq \xi_2(G) + 2 - |[\{x,y\}, \overline{U}]|.$$

Moreover, it is $\xi_2(G) \geq 2(\delta - 1)$ $(*)$. Thus, for $|[y,\overline{U}]| = 0$ we obtain

$$\begin{aligned} r_2(G) &\geq \xi_2(G) + 2 - |[\{x,y\}, \overline{U}]| = \xi_2(G) + 2 \\ &\geq \frac{1}{\delta}\left(\delta\xi_2(G) - \xi_2(G) + \xi_2(G) + 2\delta\right) \\ &\overset{(*)}{\geq} \frac{1}{\delta}\left((\delta-1)\xi_2(G) + 2\delta - 2 + 2\delta\right) \\ &\overset{(\delta\geq 2)}{\geq} \frac{1}{\delta}\left((\delta-1)\xi_2(G) + 2\delta + 2\right), \end{aligned}$$

and we are done. Analogously, in case $|[y,\overline{U}]| = 1$ we have

$$r_2(G) \geq \frac{1}{\delta}\left((\delta-1)\xi_2(G) + 2\delta - 2 + \delta\right). \tag{1}$$

If G does not fulfill the conclusion of this theorem, then (1) implies $\delta = 2$, $\xi_2(G) = 2\delta - 2$ and $r_2(G) = \xi_2(G) + 1 = 3$. Since $d(x) \geq \delta \geq 2$, it follows that U induces the path zxy in G with $|[z,\overline{U}]| \geq 1$. Therefore, $|[\{x,y\}, U \setminus \{x,y\}]| = |[\{x,y\}, z]| = 1 \leq |[z,\overline{U}]| = |[U \setminus \{x,y\}, \overline{U}]|$ and thus, G is λ_2-optimal by Lemma 2.2, a contradiction.

Hence, we may assume that $|[y,\overline{U}]| \geq 2$. If $|[y,\overline{U}]||X| \geq |X| + |Y|$, then

$$|[\{x,y\}, U \setminus \{x,y\}]| = |X| + |Y| \leq |[y,\overline{U}]||X| \overset{(\diamond)}{\leq} |[X,\overline{U}]| \leq |[U \setminus \{x,y\}, \overline{U}]|$$

and G is λ_2-optimal by Lemma 2.2, a contradiction. So assume that $|[y,\overline{U}]||X| \leq |X| + |Y| - 1$. Since $|Y| \leq |U| - |X| - 2$ and $|X| \geq \delta - 1$ it is

$$|[y,\overline{U}]| \leq \frac{|X| + |Y| - 1}{|X|} \leq \frac{|U| - 3}{|X|} \leq \frac{|U| - 3}{\delta - 1}.$$

By using this inequality we deduce

$$\begin{aligned}\xi_2(G) &\le |[\{x,y\}, V(G)\setminus\{x,y\}]| \le |U| - 2 + |[y,\overline{U}]| \\ &\le |U| - 2 + \frac{|U|-3}{\delta-1} = \frac{\delta}{\delta-1}|U| - \frac{2\delta+1}{\delta-1}.\end{aligned}$$

Since $|U| = r_2(G)$, we conclude that

$$r_2(G) \ge \frac{\delta-1}{\delta}\xi_2(G) + \frac{2\delta+1}{\delta}$$

and the proof is complete. □

The following result of Ueffing and Volkmann [7] is a direct consequence of Theorem 2.3.

Corollary 2.4 (Ueffing & Volkmann [7] 2003)**.** *Let G be a λ_2-connected and triangle-free graph with minimum degree $\delta \ge 2$. If G is not λ_2-optimal, then*

$$r_2(G) \ge \begin{cases} 2\delta - 1 & \delta \ge 3 \\ 4 & \delta = 2 \end{cases}.$$

Proof. Since $\xi_2(G) \ge 2(\delta-1)$ it follows from Theorem 2.3 that

$$r_2(G) \ge \frac{1}{\delta}\left((\delta-1)\xi_2(G) + 2\delta + 1\right) \ge \frac{1}{\delta}\left(2(\delta-1)^2 + 2\delta + 1\right) = 2\delta - 2 + \frac{3}{\delta}.$$

Because $r_2(G)$ is an integer, we conclude that $r_2(G) \ge 2\delta - 1$ for $\delta \ge 3$ and $r_2(G) \ge 4$ for $\delta = 2$. □

The graphs defined in the following example show that the bound in Theorem 2.3 is tight (see Figure 1).

Example 2.5. *For $\delta \ge 2$ and $s \ge \delta - 1$ let H_1 and H_2 be copies of the complete bipartite graphs $K_{\delta,(\delta-1)s+2}$ and $K_{s+1,\delta s}$, respectively. Join H_1 and H_2 by all possible edges between their partition sets of size δ and $s+1$. The resulting graph G is bipartite and has minimum degree δ. Furthermore, it fulfills $\lambda_2(G) = (s+1)\delta$, $\xi_2(G) = (s+1)\delta + 1$ and $r_2(G) = (\delta-1)s + 2 + \delta$. In particular, G is not λ_2-optimal. Moreover,*

$$\frac{\delta-1}{\delta}\xi_2(G) + \frac{2\delta+1}{\delta} = (\delta-1)(s+1) + 3 = r_2(G)$$

which shows that the bound in Theorem 2.3 is tight.

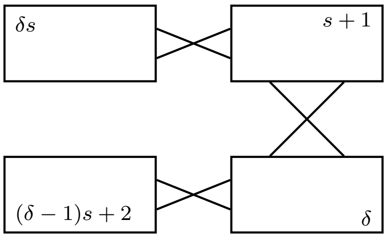

Figure 1: Graphs showing the tightness of Theorem 2.3

3 λ_3-optimality in triangle-free graphs

Ou [5] presented the following conjecture on λ_3-optimality.

Conjecture 3.1 (Ou [5]). *Let G be a connected triangle-free graph of order $n \geq 6$. If $d(u) + d(v) \geq \frac{n}{2} + 2$ for each pair u, v of non-adjacent vertices, then G is λ_3-optimal.*

This bound conjectured by Ou is slightly incorrect as we will see in Theorem 3.4. We give the correct Ore-type condition for a triangle-free graph to be λ_3-optimal. Moreover, we present examples that show the tightness of our result.

In 2002, Bonsma, Ueffing and Volkmann [1] characterized the graphs that are not λ_3-connected.

Theorem 3.2 (Bonsma, Ueffing & Volkmann [1] 2002). *A graph G is λ_3-connected if and only if $n \geq 6$ and G is not isomorphic to the net N or to any graph of the family F in Figure 2.*

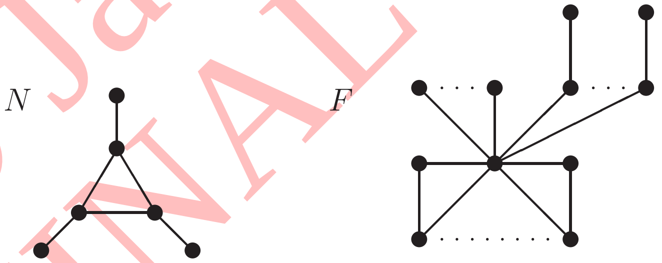

Figure 2: All graphs that are not λ_3-connected

Note that N as well as every graph in F contains a triangle. The same authors showed that the following inequality is true.

Theorem 3.3 (Bonsma, Ueffing & Volkmann [1] 2002). *If G is a λ_3-connected graph, then $\lambda_3(G) \leq \xi_3(G)$.*

Now we are able to prove the main result of this section.

Theorem 3.4. *Let G be a connected triangle-free graph of order $n \geq 6$. If $d(u) + d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 3$ for each pair u, v of non-adjacent vertices, then G is λ_3-optimal.*

Proof. Since G is not isomorphic to the net N and does not belong to the graph class F depicted in Figure 2, it follows from Theorem 3.2 that G is λ_3-connected. Thus, Theorem 3.3 yields $\lambda_3(G) \leq \xi_3(G)$.

Let $[U, \overline{U}]$ be a λ_3-cut of G with $4 \leq |U| \leq |\overline{U}|$. This implies $|U| \leq \frac{n}{2}$. Let H be a connected subgraph of order 3 of $G[U]$ such that $|[V(H), V(G) \setminus V(H)]|$ is minimal among all connected subgraphs of order 3 of $G[U]$. Note that H, as well as any other connected subgraph of G of order three, is a path on three vertices, since G is triangle-free. Let $H = xyz$. Since our goal is to apply Lemma 2.2, we have to show that

$$|[V(H), U \setminus V(H)]| \leq |[U \setminus V(H), \overline{U}]|.$$

Since G is triangle-free, x and y as well as y and z do not have common neighbors. Hence we may partition $N(H) \setminus (\overline{U} \cup V(H))$ as follows:

$$\begin{aligned}
X_0 &= \{v \in N(x) \setminus (\overline{U} \cup V(H) \cup N(z)) \colon N(v) \cap \overline{U} = \emptyset\},\\
X_1 &= \{v \in N(x) \setminus (\overline{U} \cup V(H) \cup N(z)) \colon |N(v) \cap \overline{U}| \geq 1\},\\
Y_0 &= \{v \in N(y) \setminus (\overline{U} \cup V(H)) \colon N(v) \cap \overline{U} = \emptyset\},\\
Y_1 &= \{v \in N(y) \setminus (\overline{U} \cup V(H)) \colon |N(v) \cap \overline{U}| \geq 1\},\\
Z_0 &= \{v \in N(z) \setminus (\overline{U} \cup V(H) \cup N(x)) \colon N(v) \cap \overline{U} = \emptyset\},\\
Z_1 &= \{v \in N(z) \setminus (\overline{U} \cup V(H) \cup N(x)) \colon |N(v) \cap \overline{U}| \geq 1\},\\
W_0 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H)) \colon N(v) \cap \overline{U} = \emptyset\},\\
W_1 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H)) \colon |N(v) \cap \overline{U}| = 1\},\\
W_2 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H)) \colon |N(v) \cap \overline{U}| \geq 2\}.
\end{aligned}$$

Claim 1. It is $d(v) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$ for all $v \in N(H)$.

Suppose that $v \in X_0 \cup X_1$. Based on the choice of H, we conclude that $d(v) \geq d(z)$. If $d(z) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$, we are done. Otherwise $d(z) \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$. Since v and z are not adjacent, it follows that

$$d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 3 - d(z) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

We can analogously show that the proposition is true if $v \in Z_0 \cup Z_1$ or $v \in W_0 \cup W_1 \cup W_2$.

Suppose that $v \in Y_0 \cup Y_1$. Based on the choice of H, we conclude that $d(v) \geq \max\{d(x), d(z)\}$. Since x and z are not adjacent, it follows that

$$d(v) \geq \max\{d(x), d(z)\} \geq \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

So Claim 1 is proved.

Claim 2. $X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1$ is an independent vertex set.

Suppose that $u, v \in X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1$ such that u and v are adjacent. Since G is triangle-free, their respective neighborhoods are disjoint. Furthermore, at most one of them is in W_1. It follows that

$$|U| \geq d(u) + d(v) - |[\{u,v\}, \overline{U}]| \geq d(u) + d(v) - 1 \overset{\text{(Claim 1)}}{\geq} 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 2\right) - 1 > \frac{n}{2},$$

a contradiction. So Claim 2 is proved.

Now we distinguish two cases depending on the number of vertices in X_0, Y_0, Z_0, W_0 and W_1.

Case 1. Suppose that $X_0 = Y_0 = Z_0 = W_0 = W_1 = \emptyset$. Then

$$\begin{aligned} |[V(H), U \setminus V(H)]| &= |X_1| + |Y_1| + |Z_1| + 2|W_2| \\ &\leq \sum_{v \in X_1 \cup Y_1 \cup Z_1 \cup W_2} |N(v) \cap \overline{U}| \\ &\leq |[U \setminus V(H), \overline{U}]|. \end{aligned}$$

Hence G is λ_3-optimal by Lemma 2.2.

Case 2. Suppose that $A = X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1 \neq \emptyset$. Let $B_1 = N(A) \cap N(V(H))\cap$, $B_2 = (U \cap N(A)) \setminus N[V(H)]$ and $B = B_1 \cup B_2$. Note that A is an independent set by Claim 2. Furthermore, A, B_1 and B_2 are disjoint subsets of U. Let a be an arbitrary vertex in A. Note that a has at least

$$|N(a) \cap B| \geq \left\lfloor \frac{n}{4} \right\rfloor + 2 - |[a, V(H)]| - |[a, \overline{U}]| \geq \left\lfloor \frac{n}{4} \right\rfloor - |[a, \overline{U}]| \geq \left\lfloor \frac{n}{4} \right\rfloor - 1 \qquad (2)$$

neighbors in B.

If $b \in N(a) \cap B_1$, then

$$2\left(\left\lfloor \frac{n}{4} \right\rfloor + 2\right) \leq d(a) + d(b) \leq |U| + |[\{a,b\}, \overline{U}]|.$$

Since $|U| \leq \frac{n}{2}$, it follows that $|[\{a,b\}, \overline{U}]| \geq 3$. Since G is triangle-free, $a \in W_0 \cup W_1$ implies $b \notin W_2$ and therefore $|[V(H), \{a,b\}]| \leq 3$. All in all we conclude that

$$|[V(H), \{a,b\}]| \leq 3 \leq |[\{a,b\}, \overline{U}]|.$$

If $b, b' \in N(a) \cap B_2$, then b and b' are not adjacent. It follows that

$$\begin{aligned}
2\left\lfloor \frac{n}{4} \right\rfloor + 3 &\le d(b) + d(b') \\
&\le 2(|U| - |N(a) \cap B| - |V(H)|) + |[\{b, b'\}, \overline{U}]| \\
&\le n - 2\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right) - 6 + |[\{b, b'\}, \overline{U}]| \\
&= n - 2\left\lfloor \frac{n}{4} \right\rfloor - 4 + |[\{b, b'\}, \overline{U}]|
\end{aligned}$$

and thus,

$$|[\{b, b'\}, \overline{U}]| \ge 4\left\lfloor \frac{n}{4} \right\rfloor - n + 7 \ge 4.$$

Therefore, it exists at most one vertex $b \in N(a) \cap B_2$ with $|[b, \overline{U}]| \le 1$. Let $B' = B_1 \cup B_2'$ with $B_2' = \{b \in B_2 \colon |[b, \overline{U}]| \le 1\}$. Then

$$|[V(H), \{a, b\}]| \le 2 \le |[\{a, b\}, \overline{U}]|$$

for every $b \in B'$.

If there exists a matching M of size $|A|$ connecting vertices of A and B', then

$$\begin{aligned}
|[U \setminus V(H), \overline{U}]| &\ge \sum_{ab \in M} |[\{a, b\}, \overline{U}]| + \sum_{v \in N(H) \setminus V(M)} |[v, \overline{U}]| \\
&\ge \sum_{ab \in M} |[V(H), \{a, b\}]| + \sum_{v \in N(H) \setminus V(M)} |[V(H), v]| \\
&= |[V(H), U \setminus V(H)]|
\end{aligned}$$

and thus, G is λ_3-optimal by Lemma 2.2. We distinguish two cases to show the existence of such a matching.

Subcase 2.1. Suppose that $A \setminus W_1 \neq \emptyset$. For a vertex $a \in A \setminus W_1$, (2) implies that $|B| \ge |N(a) \cap B| \ge \lfloor \frac{n}{4} \rfloor$. Thus

$$|A| \le |U| - |B| - |V(H)| \le \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 3.$$

Moreover,

$$|N(a) \cap B'| \ge |N(a) \cap B| - 1 \ge \left\lfloor \frac{n}{4} \right\rfloor - 1$$

and thus $|N(a) \cap B'| \ge |A|$ for every vertex $a \in A$. It follows that there exists a matching M of size $|A|$ connecting vertices of A and B'.

Subcase 2.2. Suppose that $A = W_1$. Then $|B| \ge |N(a) \cap B| \ge \lfloor \frac{n}{4} \rfloor - 1$ for every vertex $a \in A$. Furthermore

$$|A| \le |U| - |B| - |V(H)| \le \left\lfloor \frac{n}{2} \right\rfloor - \left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right) - 3 = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 2$$

and

$$|N(a) \cap B'| \geq |N(a) \cap B| - 1 \geq \left\lfloor \frac{n}{4} \right\rfloor - 2.$$

If $n = 4r + s$ with $0 \leq s \leq 3$, it follows that

$$|A| \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 2 < r \quad \text{and} \quad |N(a) \cap B'| \geq \left\lfloor \frac{n}{4} \right\rfloor - 2 = r - 2.$$

Hence $|N(a) \cap B'| \geq |A| - 1$ and $|N(a) \cap B| \geq |A|$ for every vertex $a \in A$. It follows that there exists a matching M of size $|A| - 1$ connecting vertices of A and B'. If M can be extended to a matching of size $|A|$ connecting vertices of A and B', then we are done. So assume that M cannot be extended in this way. Let a denote the single vertex in $a \in A \setminus V(M)$.

If $|A| \geq 2$, then $|N(a) \cap B'| \geq |A| - 1 \geq 1$. Moreover, a has at most $|A| - 1$ neighbors in B', since M cannot be extended. Furthermore, $|N(a) \cap B| \geq |A|$ implies that a has another neighbor in $B_2 \setminus B'$. Let b and b' denote these neighbors such that $b \in B_2 \setminus B'$ and $b' \in B'$ with $a'b' \in M$. Note that b and b' are not adjacent and $b' \notin W_2$, since $a \in W_1$, so b' can have at most one neighbor in $V(H)$. Then

$$\begin{aligned} 2\left\lfloor \frac{n}{4} \right\rfloor + 3 &\leq d(b) + d(b') \\ &\leq 2(|U| - |N(a) \cap B| - |V(H)|) + 1 + |[\{b, b'\}, \overline{U}]| \\ &\leq n - 2\left\lfloor \frac{n}{4} \right\rfloor - 3 + |[\{b, b'\}, \overline{U}]| \end{aligned}$$

and thus, $|[\{b, b'\}, \overline{U}]| \geq 3$. It follows that

$$|[\{a, a', b, b'\}, \overline{U}]| \geq 5 \geq |[V(H), \{a, a', b, b'\}]|.$$

If $|A| = 1$, then $A = \{a\}$ and $B' = \emptyset$. Note that $|B| \geq |A|$ and thus, $B_2 = B \neq \emptyset$. Let $b \in B_2$ be an arbitrary vertex. Then $|[b, \overline{U}]| \geq 2$ and thus,

$$|[\{a, b\}, \overline{U}]| \geq 2 \geq |[V(H), \{a, b\}]|.$$

In both cases, let $M' = M \cup \{ab\}$. Then

$$\begin{aligned} |[U \setminus V(H), \overline{U}]| &\geq \sum_{vw \in M'} |[\{v, w\}, \overline{U}]| + \sum_{v \in N(H) \setminus V(M')} |[v, \overline{U}]| \\ &\geq \sum_{vw \in M'} |[V(H), \{v, w\}]| + \sum_{v \in N(H) \setminus V(M')} |[V(H), v]| \\ &= |[V(H), U \setminus V(H)]| \end{aligned}$$

and thus, G is λ_3-optimal by Lemma 2.2. This completes the proof of the theorem. □

The following examples show that the lower bound in the Ore-type condition of the theorem above is tight.

Example 3.5. *a) Let H_1 be a copy of the complete bipartite graph $K_{r,r}$ and H_2 a copy of the complete bipartite graph $K_{r,s}$, where $s \in \{r, r+1\}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$ such that $|V_2| = s$. If $r = 2$ and $s = 3$, let $x \in V_2$. Join x by two edges to V_1 and join $V_2 \setminus \{x\}$ by a perfect matching to U_1. Otherwise join U_1 and U_2 as well as V_1 and V_2 by a matching of size r. For $s = r+1$ join the remaining vertex of V_2 to an arbitrary vertex of V_1.*

b) Let H_1 be a copy of the complete bipartite graph $K_{r,r+1}$ and H_2 a copy of the complete bipartite graph $K_{s,r+1}$, where $s \in \{r, r+1\}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$. Join U_1 and V_1 by a perfect matching.

For each $n = 4r + t$, where $r \geq 2$ and $0 \leq t \leq 3$, we have defined exactly one graph G_n. These graphs are triangle-free, have minimum degree $r+1$, and fulfill $d(u) + d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 2$ for each pair u, v of vertices. Furthermore, the edges between H_1 and H_2 form a 3-restricted edge cut of size $< \xi_3(G_n) = 3r - 1$. Hence the graphs are not λ_3-optimal.

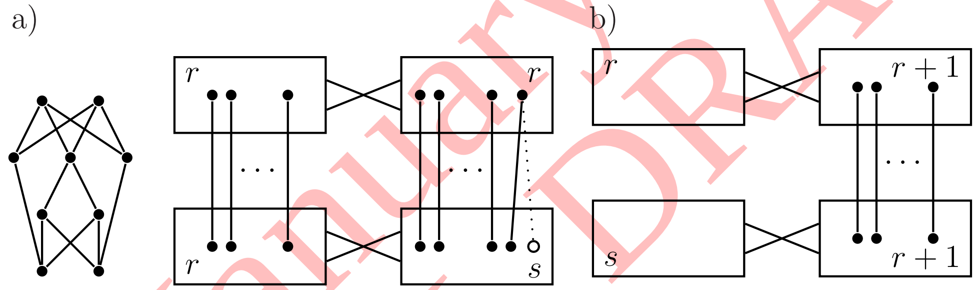

Figure 3: Graphs showing the tightness of Theorem 3.4

4 λ_k-optimality in triangle-free graphs

In 2005, Zhang and Yuan [10] proved that, except for the class of flowers, graphs with minimum degree greater or equal than $k - 1$ are λ_k-connected. Moreover, for the same class of graphs they showed that $\lambda_k(G) \leq \xi_k(G)$. (A graph G with $|V(G)| \geq 2k$ is called a *flower* if it contains a cut vertex u such that every component of $G - u$ has order at most $k - 1$.)

Theorem 4.1 (Zhang & Yuan [10] 2005)**.** *Let G be a connected graph not isomorphic to a flower and k a positive integer with $k \leq \delta(G) + 1$. Then, G is λ_k-connected and $\lambda_k(G) \leq \xi_k(G)$.*

Furthermore, for graphs that are not λ_k-optimal, in 2007 the same authors gave a lower bound on the order of their k-fragments.

Theorem 4.2 (Zhang & Yuan [11] 2007)**.** *Let G be a λ_k-connected graph with minimum degree $\delta(G)$. If $\lambda_k(G) < \xi_k(G)$, then $r_k(G) \geq \max\{k+1, \delta(G)-k+1\}$.*

Our first result of this section, namely Theorem 4.6, will present a new lower bound on the order of these k-fragments, which is a generalization of the earlier result from Ueffing and Volkmann [7] in Corollary 2.4 for the case $k = 2$. In order to prove this result, we make use of a variation of a well-known result from Turán.

Theorem 4.3 (Turán [6] 1941)**.** *Let G be a graph of order n without a $(p+1)$-clique. Then*

$$2|E(G)| \leq \frac{(p-1)|V(G)|^2}{p}.$$

More precisely, the graph $K_{n_1,n_2,\dots,n_p}$ with $n_1+n_2+\dots+n_p = |V(G)|$ and $|n_i - n_j| \leq 1$ is the unique extremal graph without a $(p+1)$-clique and the maximum number of edges.

Remark 4.4. *Let G be a connected graph with an independent vertex set $S \subset G$ of order $\left\lceil \frac{|V(G)|}{2} \right\rceil$. Then there exists a vertex $v \in S$ such that $G - v$ is connected.*

Proof. Consider a spanning tree T of G. Then, as S is an independence set of order $\left\lceil \frac{|V(G)|}{2} \right\rceil$, there exists a vertex $v \in V(T)$ such that v is a leaf in T, i.e. $d_T(v) = 1$, which means $T - v$ is connected, and hence, $G - v$ is connected. $\square$

We call a graph G *bipartite and balanced* if G is a bipartite graph with partite sets $V_1(G)$ and $V_2(G)$ such that $||V_1(G)| - |V_2(G)|| \leq 1$.

Lemma 4.5. *Let G be a connected and triangle-free graph of order n and $W \subset G$ be connected with $|V(W)| = k$ and the maximum number of edges in G among all connected subgraphs of G of order k. Then*

$$|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor + |E(W)| - \left\lfloor \frac{k^2}{4} \right\rfloor.$$

Proof. For $k = 1$ the lemma is a direct consequence of Theorem 4.3. So let $k \geq 2$ and let G^* be a bipartite and balanced graph of order $n = |V(G)|$ with partite sets V_1 and V_2 fulfilling $|V_1| = \left\lceil \frac{n}{2} \right\rceil$ and $|V_2| = \left\lfloor \frac{n}{2} \right\rfloor$. Furthermore, let W^* be an induced bipartite and balanced subgraph of G^* with partite sets $W_1 \subset V_1$ and $W_2 \subset V_2$ fulfilling $|W_1| = \left\lceil \frac{k}{2} \right\rceil$ and $|W_2| = \left\lfloor \frac{k}{2} \right\rfloor$. As the triangle-free graph with the maximum number of edges is a bipartite and balanced graph by Theorem 4.3, we define the edges of G^* such that $|E(G^*)| = |E(G)|$ and $|E(W^*)|$ is minimum. From this construction it follows that

$$|E(G^*)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor + |E(W^*)| - \left\lfloor \frac{k^2}{4} \right\rfloor .$$

If $|E(W^*)| \leq |E(W)|$, we have

$$|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor + |E(W)| - \left\lfloor \frac{k^2}{4} \right\rfloor$$

and the lemma holds. Thus, let

$$|E(W)| < |E(W^*)|. \tag{3}$$

Hence, $|E(W^*)| \geq 1$, $|E(G^* - W^*)| = \left\lfloor \frac{(n-k)^2}{4} \right\rfloor$ and

$$|[V(G^* - W^*), V(W^*)]| = \left\lceil \frac{k(n-k)}{2} \right\rceil \tag{4}$$

by the choice of W^*. Furthermore, by Theorem 4.3 we have

$$|E(G - W)| \leq |E(G^* - W^*)| \tag{5}$$

as $G^* - W^*$ is a complete bipartite and balanced graph. Since

$$\begin{aligned} &|E(G^* - W^*)| + |[V(G^* - W^*), V(W^*)]| + |E(W^*)| \\ &\qquad = |E(G - W)| + |[V(G - W), V(W)]| + |E(W)|, \end{aligned}$$

it follows from (3), (4) and (5) that $\left\lceil \frac{k(n-k)}{2} \right\rceil + 1 \leq |[V(G - W), V(W)]|$.

Case 1. Suppose that k is even. As $\frac{k(n-k)}{2} + 1 \leq |[V(G - W), V(W)]|$, there exists a vertex $u \in V(G) \setminus V(W)$ such that $|N(u) \cap V(W)| \geq \frac{k}{2} + 1$. Since G is triangle-free, $N(u) \cap V(W)$ is an independent vertex set. Thus, by Remark 4.4 there exists a vertex $w \in N(u) \cap V(W)$ such that $W - w$ is connected. Consider the induced graph H in G with vertices $V(H) = (V(W) - \{w\}) \cup \{u\}$. The graph H is connected, has order k and size $|E(W)| < |E(H)|$ (as $|N(w) \cap V(W)| \leq \frac{k}{2} - 1 < \frac{k}{2} \leq |N(u) \cap (V(W) \setminus \{w\})|$), contradicting the choice of W.

Case 2. Suppose that k is odd. As $\left\lceil \frac{k(n-k)}{2} \right\rceil + 1 \leq |[V(G - W), V(W)]|$, there exists a vertex $u \in V(G) \setminus V(W)$ such that $|N(u) \cap V(W)| \geq \frac{k+1}{2}$. Since G is triangle-free, $N(u) \cap V(W)$ is an independent vertex set. Therefore, by Remark

4.4 there exists a vertex $w \in N(u) \cap V(W)$ such that $W - w$ is connected. Consider the induced graph H in G with vertices $V(H) = (V(W) - \{w\}) \cup \{u\}$. H is a connected graph in G of order k and size $|E(W)| \leq |E(H)|$ (as $|N(w) \cap V(W)| \leq k - \frac{k+1}{2} = \frac{k-1}{2} \leq |N(u) \cap (V(W) \setminus \{w\})|$). By the choice of W it follows that $|E(W)| = |E(H)|$, which means $|N(w) \cap V(W)| = \frac{k-1}{2} = |N(u) \cap (V(W) \setminus \{w\})|$.

Subcase 2.1. There exists a vertex $w' \in N(u) \cap V(W)$ such that

$$|N(w') \cap V(W)| \leq \frac{k-1}{2} - 1.$$

Consider the induced graph H in G with vertices $V(H) = (V(W) - \{w'\}) \cup \{u\}$. H has order k and size $|E(W)| < |E(H)|$ (as $|N(w') \cap V(W)| \leq \frac{k-1}{2} - 1 < \frac{k-1}{2} = |N(u) \cap (V(W) \setminus \{w'\})|$). Observe that H is connected as $V(W) = (N(u) \cap V(W)) \cup (N(w) \cap V(W))$, contradicting the choice of W.

Subcase 2.2. It is $|N(w') \cap V(W)| \geq \frac{k-1}{2}$ for all $w' \in N(u) \cap V(W)$. Thus, W has at least $(\frac{k+1}{2})(\frac{k-1}{2}) = \frac{k^2-1}{4}$ edges, and must therefore be a complete bipartite and balanced subgraph of G. Due to Theorem 4.3, it is $|E(W)| = \left\lfloor \frac{k^2}{4} \right\rfloor$ and the proof is complete. □

Theorem 4.6. *Let G be a λ_k-connected and triangle-free graph with minimum degree $\delta(G)$ and $\lambda_k(G) \leq \xi_k(G)$. If G is not λ_k-optimal, then $r_k(G) \geq \max\{k+1, 2\delta(G) - k + 1\}$.*

Proof. Since G is not λ_k-optimal it is $\lambda_k(G) < \xi_k(G)$. Let U be a λ_k-atom of G. If $|U| = k$, then $\xi_k(G) \leq |[U, \overline{U}]| = \lambda_k(G)$, which is a contradiction. So assume that $r_k(G) = |U| \geq k + 1$. Let W be a connected subgraph of $G[U]$ with $|V(W)| = k$ and the maximum number of edges among all connected subgraphs of $G[U]$ of order k. By Lemma 4.5 we have

$$|E(G[U])| \leq \left\lfloor \frac{r_k^2}{4} \right\rfloor + |E(W)| - \left\lfloor \frac{k^2}{4} \right\rfloor.$$

As $\lambda_k(G) = |[U, \overline{U}]| = \sum_{v \in U} d_G(v) - 2|E(G[U])|$, it follows that

$$\lambda_k \geq \sum_{v \in U} d_G(v) - 2\left(\left\lfloor \frac{r_k^2}{4} \right\rfloor + |E(W)| - \left\lfloor \frac{k^2}{4} \right\rfloor\right).$$

Since

$$\xi_k(G) \leq |[V(W), \overline{V(W)}]| = \sum_{v \in W} d_G(v) - 2|E(W)|,$$

and $\lambda_k(G) < \xi_k(G)$, we have

$$\sum_{v \in U \setminus V(W)} d_G(v) - 2\left\lfloor \frac{r_k^2}{4} \right\rfloor + 2\left\lfloor \frac{k^2}{4} \right\rfloor < 0. \tag{6}$$

Case 1. Suppose that k is even. It follows from (6) that

$$\delta(G)(r_k - k) - \frac{r_k^2}{2} + \frac{k^2}{2} < 0. \tag{7}$$

Case 2. Suppose that k is odd.

Subcase 2.1. Suppose that r_k is odd. Since $(k+1)$ and $(k-1)$ are both even numbers, $\frac{(k+1)}{2}\frac{(k-1)}{2} = \frac{k^2-1}{4}$ is an integer. Analogously, $\frac{r_k^2-1}{4}$ is an integer, and thus, it follows from (6) that again

$$\delta(G)(r_k - k) - \frac{r_k^2 - 1}{2} + \frac{k^2 - 1}{2} = \delta(G)(r_k - k) - \frac{r_k^2}{2} + \frac{k^2}{2} < 0. \tag{8}$$

The inequalities (7) or (8) directly lead to

$$(r_k - k)\left(\frac{r_k}{2} - \delta(G) + \frac{k}{2}\right) > 0.$$

Since $r_k(G) \geq k + 1$, we have $r_k(G) > 2\delta(G) - k$ proving Theorem 4.6 for both Case 1 and Subcase 2.1.

Subcase 2.2. Suppose that r_k is even. Like above, $\frac{k^2-1}{4}$ is an integer. Together with (6) we obtain

$$\delta(G)(r_k - k) - \frac{r_k^2}{2} + \frac{k^2 - 1}{2} < 0. \tag{9}$$

If $r_k \geq k + 3$ this leads to $r_k(G) > 2\delta(G) - k - \frac{1}{3}$, yielding $r_k(G) \geq 2\delta(G) - k$. Since $2\delta(G) - k$ is an odd number, and r_k is even, we obtain

$$r_k(G) \geq 2\delta(G) - k + 1.$$

In the remaining case that $r_k = k + 1$ for an odd k, the inequality (9) yields

$$\delta(G) - \frac{r_k^2}{2} + \frac{(r_k - 1)^2 - 1}{2} = \delta(G) - r_k < 0.$$

Therefore, it is $r_k \geq \delta(G) + 1$, $k \geq \delta(G)$, $r_k + k \geq 2\delta(G) + 1$, and the proof is complete. □

The following example shows that the bound given in Theorem 4.6 is tight.

Example 4.7. *For* $\delta \geq 2$ *and* $s > \frac{k^2}{2} - \frac{1}{2}$ *when* k *is odd and* $s > \frac{k^2}{2}$ *when* k *is even, let* H_1*,* H_2 *be two copies of the complete bipartite graph* $K_{s-k+1,s}$*. Let* U_i*,* V_i *be the partition sets of* H_i *for* $i = 1, 2$*. If we join* V_1 *and* V_2*, i.e. the independence sets of cardinality* s*, by* $k-1$ *perfect matchings, the resulting graph* G *is bipartite and* s*-regular. Furthermore, it fulfills* $\lambda_k(G) \leq ks - s$*,* $\xi_k(G) \geq ks - \frac{k^2}{2} + \frac{1}{2}$ *if* k *is odd, and* $\xi_k(G) \geq ks - \frac{k^2}{2}$ *if* k *is even. Therefore,* G *fulfills* $\lambda_k(G) < \xi_k(G)$*. Moreover,* $r_k(G) = 2s - k + 1$*, which shows that the bound in Theorem 4.6 is tight.*

In 2009 Yuan, Liu and Wang [9] showed the λ_k-optimality for bipartite graphs with high minimum degree.

Theorem 4.8 (Yuan, Liu & Wang [9] 2009)**.** *Let* G *be a bipartite graph of order* $n \geq 2k$*. If* $\delta(G) \geq \frac{n+2k}{4}$*, then* G *is* λ_k*-optimal.*

By using Theorem 4.6 it is now very easy to prove the following generalization of this result.

Corollary 4.9. *Let* G *be a connected and triangle-free graph or order* $n \geq 2k$*. If*

$$\delta(G) \geq \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right),$$

then G *is* λ_k*-optimal.*

Proof. Since $k \leq \delta(G) + 1$, by Theorem 4.1 it follows that $\lambda_k(G) \leq \xi_k(G)$. Therefore, as $r_k(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, the inequality holds by Theorem 4.6. $\square$

Similar to the proof of Theorem 4.6, we can give a lower bound on the order of the k-fragments of a λ_k-optimal graph that are larger than k.

Theorem 4.10. *Let* G *be a* λ_k*-optimal and triangle-free graph. If* U *is a* λ_k*-fragment of* G *with* $|U| \geq k+1$*, then* $|U| \geq 2\delta(G) - k$*.*

Proof. The proof is analogue to the proof of Theorem 4.6 except that $\lambda_k(G) = \xi_k(G)$ is used here instead of $\lambda_k < \xi_k(G)$. $\square$

As a consequence of this result, we obtain Corollary 4.12, which is a generalization of this earlier result from Yuan, Liu and Wang [9] in 2009.

Theorem 4.11 (Yuan, Liu & Wang [9] 2009)**.** *Let* G *be a bipartite graph of order* $n \geq 2k$*. If* $\delta(G) \geq \frac{n+2k+3}{4}$*, then* G *is super-*λ_k*.*

Corollary 4.12. *Let G be a triangle-free graph of order $n \geq 2k$. If*

$$\delta(G) \geq \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor + k + 1\right),$$

then G is super-λ_k.

Proof. By Corollary 4.9, G is λ_k-optimal. Suppose by contradiction that G is not super-λ_k. Then, there exists a k-fragment U such that $|U| \geq k+1$ and $|\overline{U}| \geq k+1$. We may suppose $|U| \leq |\overline{U}|$ which means $|U| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore, combining the fact $\delta(G) \geq \frac{1}{2}(\left\lfloor \frac{n}{2} \right\rfloor + k + 1)$ with Theorem 4.10, it follows that $|U| \geq 2\delta(G) - k \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, contradicting $|U| \leq \left\lfloor \frac{n}{2} \right\rfloor$. ☐

The following upper bound for ξ_k in regular graphs is trivial. (Take a tree of order k in G and count the outgoing edges.)

Observation 4.13. *If G is a δ-regular graph, then $\xi_k(G) \leq k\delta - 2(k-1)$.*

Together with Corollary 4.12 this observation gives a lower bound on the order of k-fragments in terms of ξ_k for regular and triangle-free graphs that are not λ'-optimal.

Corollary 4.14. *Let G be a λ_k-connected, δ-regular and triangle-free graph with $\lambda_k(G) \leq \xi_k(G)$. If G is not λ_k-optimal, then $r_k(G) \geq \frac{2}{k}\left(\xi_k(G) - 1\right) + 5 - k$.*

Proof. Since $\xi_k(G) \leq k\delta - 2(k-1)$, it follows that $\frac{\xi_k(G)+2k-2}{k} \leq \delta$. Then, by Theorem 4.6, $r_k(G) \geq 2\delta(G) - k + 1 \geq 2\left(\frac{\xi_k(G)+2k-2}{k}\right) - k + 1 = \frac{2}{k}\left(\xi_k(G) - 2\right) + 5 - k$. ☐

Finally, we like to give the following conjecture, which is a generalization of Theorem 4.6.

Conjecture 4.15. *Let G be a λ_k-connected graph with clique number $\omega(G) \leq p$, where $p \geq 2$, minimum degree $\delta(G)$ and $\lambda_k(G) \leq \xi_k(G)$. If G is not λ_k-optimal, then $r_k(G) \geq \max\{k+1, \frac{p}{p-1}\delta(G) - k + \frac{1}{p-1}\}$.*

The following example shows that the bound presented in the conjecture above is best possible.

Example 4.16. *Let $k = p-1$. We consider the complete p-partite graph with partition sets $V_i = \{x_i, y_i\}$ for $1 \leq i \leq p$, and remove the edges $x_i y_{i+1}$ for $1 \leq i \leq p-1$, as well as the edge $x_p y_1$. The resulting graph G has minimum degree $\delta = 2p-3$. Hence, $\xi_k = k(\delta - (k-1)) = (p-1)^2$. Moreover, $\lambda_k \leq p(p-2) < \xi_k$ (take $G[\{x_1, x_2, \ldots, x_p\}]$), which means G is not λ_k-optimal. Furthermore, $r_k = p = \frac{p}{p-1}\delta - k + \frac{1}{p-1}$.*

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