A general method in the theory of domination in graphs

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Abstract

In this paper we present a method to derive inequalities involving various domination parameters in graphs. As an application we determine several lower bounds for these domination parameters in trees in terms of the order and the number of leaves. Finally, we characterize the classes of extremal trees.

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1 Terminology

In this paper we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. The number of vertices $|V|$ is called the *order* of G and is denoted by $n(G)$. The subgraph induced by a subset X of V is denoted by $G[X]$. For two distinct vertices u and v the *distance* $d(u, v)$ between u and v is the length of a shortest path between u and v. If X and Y are two disjoint subsets of V, then the distance between X and Y is defined as $d(X, Y) = \min\{d(x, y) : x \in X,\ y \in Y\}$.

The *open k-neighborhood* $N^k(X)$ of a subset $X \subseteq V$ is the set of vertices in $V \setminus X$ of distance at most k from X and the *closed k-neighborhood* is defined by $N^k[X] = N^k(X) \cup X$. If $X = \{v\}$ is a single vertex, then we denote the (closed) k-neighborhood of v by $N^k(v)$ ($N^k[v]$, respectively). The (closed) 1-neighborhood of a vertex v or a set X of vertices is usually denoted by $N(v)$ or $N(X)$, respectively ($N[v]$ or $N[X]$, respectively).

Now let U be an arbitrary subset of V and $u \in U$. We say that v is a *private k-neighbor of u with respect to U* if $d(u, v) \le k$ and $d(u', v) > k$ for all $u' \in U \setminus \{u\}$, that is $v \in N^k[u] \setminus N^k[U \setminus \{u\}]$. The *private k-neighborhood of u with respect to U* will be denoted by $PN^k[u, U]$ ($PN^k[u]$ if $U = V$).

For a vertex $v \in V$ we define the *degree* of v as $d(v) = |N(v)|$. A vertex of degree one is called a *leaf* and the neighbor of a leaf is its *support vertex*.

We define a *matching* of a graph G to be a subset $M \subseteq E$ of edges which are pairwise not incident. A *perfect matching* is a matching of size $|V|/2$.

If we remove an edge xy in a graph G and add a new vertex z as well as the edges xz and zy, we say that the edge xy has been *subdivided*.

A set $D \subseteq V$ of vertices is said to be a *(connected) distance-k dominating set* of G if the distance between each vertex $u \in V \setminus D$ and D is at most k (and D induces a connected graph in G). The minimum cardinality of a (connected) distance-k dominating set in G is the *(connected) distance-k domination number* of G, denoted by $\gamma^k(G)$ ($\gamma_c^k(G)$, respectively). Note that the distance-1 domination number $\gamma^1(G)$ is the usual *domination number* $\gamma(G)$.

A set $D \subseteq V$ of vertices is defined to be a *distance-k total dominating set* of G if every vertex in V is within distance k from some vertex of D other than itself. The minimum cardinality among all distance-k total dominating sets of G is called the *distance-k total domination number* of G and is denoted by $\gamma_t^k(G)$.

For $x \in X \subseteq V$, if $PN^k[x] \neq \emptyset$, the vertex x is said to be *k-irredundant in X*. A set X containing only k-irredundant vertices is called *k-irredundant*. The *k-irredundance number* of G, denoted by $ir^k(G)$, is the minimum cardinality taken over all maximal k-irredundant sets of G.

We define a set $D \subseteq V$ of vertices to be a *(connected) p-dominating set* of G if every vertex outside of D has at least p neighbors in D (and $G[D]$ is connected). The minimum cardinality among all (connected) p-dominating sets is called the *(connected) p-dominating number* of G, denoted by $\gamma_p(G)$ ($\gamma_{p,c}(G)$, respectively).

An *(connected) end-dominating set* of G is a dominating set $D \subseteq V$ which contains all leaves of G (and $G[D]$ is connected). We define the *(connected) end-dominating number* of G, denoted by $\gamma_e(G)$ ($\gamma_{e,c}(G)$, respectively), as the minimum cardinality taken over all (connected) end-dominating sets of G.

For $k \in N$, we define a distance-k dominating set $D \subseteq V$ of G to be a *distance-k weakly connected dominating set* of G if the subgraph $G[N^k[D]]_w = (N^k[D], E_w)$ weakly induced by $N^k[D]$ is connected, where E_w is the set of edges which are incident with at least one vertex in $N^{k-1}[D]$. The *distance-k weakly connected domination number* is the minimum cardinality among all distance-k weakly connected dominating sets of G and will be denoted by $\gamma_{wc}^k(G)$.

A dominating set $D \subseteq V$ of G is a *paired dominating set* if the induced graph $G[D]$ has a perfect matching. The minimum cardinality of all paired dominating sets is the *paired dominating number* of G, denoted by $\gamma_{pr}(G)$.

For any integer $p \geq 2$, we define a set $D \subseteq V$ to be a *p-tuple dominating set* of G if $|N[x] \cap D| \geq p$ for every vertex x of G. The *p-tuple dominating number* of G, denoted by $\gamma_{\times p}(G)$, is the minimum cardinality taken over all p-tuple dominating sets. In the special case that $p = 2$, we use the terms *double dominating set* and *double dominating number*.

A set $D \subseteq V$ is a *restrained dominating set* if every vertex outside of D is adjacent to a vertex in D and a vertex outside of D other than itself. Let $\gamma_r(G)$ (the *restrained dominating number*) denote the size of a smallest restrained dominating set.

A set $D \subseteq V$ is a *weak dominating set* if every vertex x not in D is adjacent to a vertex $y \in D$ with $d(y) \le d(x)$. The *weak domination number* of G, denoted by $\gamma_w(G)$, is the minimum size of a weak dominating set of G.

Throughout this article the above properties of a k-distance dominating set will be denoted by the corresponding index of γ (e.g. the property 'p-tuple dominating set' will be denoted by '$\times p$').

2 Introduction

Let $G = (V, E)$ be a graph and let $\mathcal{P}$ and $\mathcal{Q}$ be properties defined on sets of vertices. Suppose that V has property $\mathcal{Q}$. Then for every subset $S \subseteq V$ there is a set $S' \subseteq V$ such that $S \subseteq S'$ and S' has property $\mathcal{Q}$. Assume that there is at least one subset of V having property $\mathcal{P}$. Suppose we are interested in finding subsets of minimum cardinality of V having property $\mathcal{P}$ or $\mathcal{Q}$. We denote by $\phi_{\mathcal{P}}(G)$ and $\phi_{\mathcal{Q}}(G)$ the minimum cardinality of a subset $S \subseteq V$ with property $\mathcal{P}$ and $\mathcal{Q}$, repectively.

Let $\mathfrak{P}_{\min}$ denote the set of subsets $D \subseteq V$ that have property $\mathcal{P}$ and minimum cardinality. Furthermore, let $\mathfrak{Q}$ denote the set of subsets $D \subseteq V$ that have property $\mathcal{Q}$. We define

$$v_{\mathcal{Q}}(\mathcal{P}) = \max_{D \in \mathfrak{P}_{\min}} \min_{D \subseteq D' \in \mathfrak{Q}} \{|D'| - |D|\} \tag{1}$$

to be the number of vertices that are required in worst case to be added to a set $D \in \mathfrak{P}_{\min}$ in order to construct a set $D' \in \mathfrak{Q}$ such that $D \subseteq D'$. It follows that

$$\phi_{\mathcal{Q}}(G) \le \phi_{\mathcal{P}}(G) + v_{\mathcal{Q}}(\mathcal{P}). \tag{2}$$

To construct useful and nontrivial inequalities involving $\phi_{\mathcal{Q}}$ and $\phi_{\mathcal{P}}$, we need properties $\mathcal{P}$ and $\mathcal{Q}$ which are in some way related such that it is possible to construct a superset with property $\mathcal{Q}$ from a set with property $\mathcal{P}$.

For example, this is the case when we compare different types of domination parameters. For instance, we can analyze relations between different distance-k domination or p-domination parameters, where some properties like 'connected', 'total', 'paired' or 'weakly connected' are added.

Now let $\mathcal{Q}'$ be a superhereditary property, that is a property such that the following holds: If $S \subseteq V$ is a set with property $\mathcal{Q}'$, every superset $S \subseteq S' \subseteq V$ of S has property $\mathcal{Q}'$. We consider the following special case: Let G be connected and let $\mathcal{Q}$ be a property on sets of vertices such that $S \subseteq V$ has property $\mathcal{Q}$ if and only if it has property $\mathcal{Q}'$ and $G[S]$ is connected. Then we can approach $\phi_{\mathcal{Q}}(G)$ in terms of $\phi_{\mathcal{P}}(G)$ by determining an upper bound for $v_{\mathcal{Q}}(\mathcal{P})$ in the following way. In the first step, we expand D to a superset D' with property $\mathcal{Q}'$ and in the second step, we expand D' to a superset D'' such that $G[D'']$ is connected. Since $\mathcal{Q}'$ is superhereditary, the set D'' has property $\mathcal{Q}$. The number of vertices needed for the expansion of D' to D'' can be approached as follows. We look for an upper bound for number of components of $G[D']$ and for the number of vertices we need to connect these components.

Using this technique (with $\mathcal{P}$ as the property of being maximal irredundant and $\mathcal{Q}'$ as the property of being connected distance-k dominating), the authors proved in a previous paper [7] the following inequalities.

Theorem 2.1 (Hansberg, Meierling & Volkmann [7] 2007)**.** *If G is a connected graph, then*

$$\gamma_c^k(G) \leq \frac{5k+1}{2} ir^k(G) - 2k.$$

Theorem 2.2 (Hansberg, Meierling & Volkmann [7] 2007)**.** *If T is a tree, then*

$$\gamma_c^k(T) \leq (2k+1) ir^k(T) - 2k.$$

These results imply the following propositions.

Corollary 2.3 (Favaron & Kratsch [6] 1991)**.** *If G is a connected graph, then*

$$\gamma_c(G) \leq 3ir(G) - 2.$$

Corollary 2.4 (Hansberg, Meierling & Volkmann [7] 2007)**.** *If T is a tree with n_1 leaves, then*

$$ir^k(G) \geq \frac{|V(T)| - kn_1 + 2k}{2k+1}.$$

Moreover, they conjectured the following.

Conjecture 2.5 (Hansberg, Meierling & Volkmann [7] 2007)**.** *If G is a connected graph, then*

$$\gamma_c^k(G) \leq (2k+1) ir^k(G) - 2k.$$

Further, if $b(G)$ is a lower bound for $\phi_{\mathcal{Q}}(G)$, we can combine it with inequality (2) to derive the inequality chain

$$b(G) \leq \phi_{\mathcal{Q}}(G) \leq \phi_{\mathcal{P}}(G) + v_{\mathcal{Q}}(\mathcal{P}). \tag{3}$$

Solving (3) for $\phi_{\mathcal{P}}(G)$, we arrive at

$$\phi_{\mathcal{P}}(G) \geq b(G) - v_{\mathcal{Q}}(\mathcal{P}), \tag{4}$$

a lower bound for $\phi_{\mathcal{P}}(G)$.

If we consider a property $\mathcal{P}$ that implies the superhereditary property $\mathcal{Q}'$, we can omit the first step in the construction of D''. We shall illustrate this in the next section by proving several inequalities for the connected distance-k dominating number γ_c^k (i.e., $\mathcal{Q}'$ is the property of being distance-k dominating) in terms of the parameters γ^k, γ_t^k, γ_{wc}^k, γ_{pr}^k and γ_e^k for $k \geq 1$ and $\gamma_{\times p}$ for $k = 1$. In particular, if G is a tree, several lower bounds $b(G)$ for the last mentioned parameters are known, which will lead us to interesting inequalities we will present in Section 3.4.

3 A method in domination

In this section we describe the method used in this paper to derive various bounds for several domination parameters.

Let $G = (V, E)$ be a connected graph and let $k \geq 1$ be an integer. Furthermore, let $\mathcal{X}$ be a property defined on sets of vertices. In terms of Section 2, $\mathcal{P}_{\mathcal{X}}$ is the property of being "distance-k dominating and additionally having property $\mathcal{X}$"; $\mathcal{Q}'$ is the property of being "distance-k dominating" and $\mathcal{Q}$ is the property of being "distance-k dominating and connected". Let D be a set of vertices of G with property $\mathcal{P}_{\mathcal{X}}$, that is, let D be a vertex-set that is distance-k dominating and has property $\mathcal{X}$.

As described in the previous section, we firstly look for a relationship between $\gamma_{\mathcal{P}_{\mathcal{X}}} = \gamma_{\mathcal{X}}^k$ and $\gamma_{\mathcal{Q}} = \gamma_c^k$. More precisely, we shall calculate the number $v_{\mathcal{Q}}(\mathcal{P}_{\mathcal{X}}) = v_c^k(\mathcal{X})$ for various parameters $\mathcal{X}$ in the following way. Using property $\mathcal{P}_{\mathcal{X}}$, we determine an upper bound $c^k(\mathcal{X})$ for the number of components of $G[D]$ and calculate $v_c^k(\mathcal{X})$ afterwards. This yields

$$\gamma_c^k = \gamma_{\mathcal{Q}} \leq \gamma_{\mathcal{P}_{\mathcal{X}}} + v_c^k(\mathcal{X}) = \gamma_{\mathcal{X}}^k + v_c^k(\mathcal{X}). \tag{5}$$

Now let G be a tree on n vertices. Note that

$$\gamma_{\mathcal{X}}^k(G) \geq n - kn_1, \tag{6}$$

where n_1 is the number of leaves of G and that any connected end-dominating set of G contains all vertices of G. Using (5) and (6), we obtain

$$\gamma_{\mathcal{X}}^k(G) \geq \gamma_c^k(G) - v_c^k(\mathcal{X}) \geq n - kn_1 - v_c^k(\mathcal{X}). \tag{7}$$

Note that if the dominating set D we started with is an end-dominating set, that is, $\mathcal{X}$ is the property of being "end-dominating", we obtain a better lower bound

$$\gamma_{\mathcal{X}}^k(G) \geq \gamma_c^k(G) - v_c^k(\mathcal{X}) \geq n - v_c^k(\mathcal{X}). \tag{8}$$

In the last section, we characterize all trees for which the lower bounds (7) and (8) are sharp. This can be done in two steps. Observe that G is a tree with $\gamma_{\mathcal{X}}^k(G) = n - kn_1$ if and only if G is a tree whose end-edges have been $(k-1)$-times subdivided. Using this observation and our construction method, we can analyze which trees fulfill (5) with equality.

3.1 Upper bounds for $c^k(\mathcal{X})$

Lemma 3.1. *Let $G = (V, E)$ be a connected graph, let L be the set of leaves of G and let S be the set of support vertices of G. Furthermore, let $\mathcal{X}$ be a property defined on sets of vertices and let D be a distance-k dominating set of minimum cardinality. Then*

(a) $c^k(\mathcal{X}) \leq |D|$ for every property $\mathcal{X}$;

(b) $c^1(t) \leq \frac{|D|}{2}$;

(c) $c^k(pr) \leq \frac{|D|}{2}$;

(d) $c^1(\times p) \leq \frac{|D|}{p}$ and if G is a tree, then

$$c^1(\times 2) \leq |S| + \frac{1}{2}(|D| - |S| - |L|);$$

(e) $c^k(e) \leq |D| - |L_1|$, where $L_1 \subseteq L$ is the set of leaves of G that have a neighbor in D.

Proof. Propositions (a)-(c), (e) and the first part of (d) are immediate by the definition of the corresponding property. It remains to show the second part of (d).

Let G be a tree. Since D is a double dominating set, all leaves as well as all support vertices of G are in D. So $G[D]$ has at most

$$|S| + \left\lfloor \frac{1}{2}(|D| - |S| - |L|) \right\rfloor$$

components which completes the proof of this lemma. □

3.2 Upper bounds for $v_c^k(\mathcal{X})$

Lemma 3.2. *Let $G = (V, E)$ be a connected graph, let L be the set of leaves of G and let S be the set of support vertices of G. Furthermore, let $\mathcal{X}$ be a property defined on sets of vertices and let D be a distance-k dominating set of minimum cardinality. Then*

(a) $v_c^k(\mathcal{X}) \leq 2k(c^k(\mathcal{X}) - 1)$ *for every property* $\mathcal{X}$*;*

(b) $v_c^k(t) \leq (k-1)\left\lceil \frac{1}{2}c^k(t)\right\rceil + 2k\left(\left\lfloor \frac{1}{2}c^k(t)\right\rfloor - 1\right)$*;*

(c) $v_c^k(wc) \leq (2k-1)(c^k(wc) - 1)$*;*

(d) $v_c^k(pr) \leq 2k(c^k(pr) - 1)$*;*

(e) $v_c^1(\times p) \leq 2(c^1(\times p) - 1)$ *and if G is a tree, then*

$$v_c^1(\times 2) \leq c^1(\times 2) - 1;$$

(f) $v_c^k(e) \leq 2k(c^k(e) + |S \setminus D| - |L \setminus L_1| - 1)$*, where $L_1 \subseteq L$ is the set of leaves of G that have a neighbor in D.*

Proof. Recall the following.

(i) The graph G is connected. This implies that for every component G_1 of $G[D]$ there is a component G_2 of $G[D]$ in distance at most $2k + 1$.

(ii) If $\mathcal{X} = t$ and x is an arbitrary vertex of D, there exists a vertex y of D such that the distance between x and y is at most k. Therefore x and y can be joined to one component by adding at most $k - 1$ vertices to D.

(iii) If $\mathcal{X} = wc$, the subgraph $(N^k[D], E_w)$ is connected. This implies that for every component G_1 of $G[D]$ there is a component G_2 of $G[D]$ in distance at most $2k$.

We now proceed with the proof. Inductive application of observation (i) yields propositions (a), (d) and the first part of (e). It remains to show propositions (b), (c), the second part of (e) and (f).

(b) Suppose that $\mathcal{X} = t$. Using (i) and (ii), we conclude that

$$v_c^k(t) \leq (k-1)\left\lceil \frac{1}{2}c^k(t)\right\rceil + 2k\left(\left\lfloor \frac{1}{2}c^k(t)\right\rfloor - 1\right).$$

(c) Suppose that $\mathcal{X} = wc$. By (iii), it follows that

$$v_c^k(wc) \le (2k-1)(c^k(wc)-1).$$

(e) Suppose that $\mathcal{X} = \times 2$ and let G be a tree. Observe that D contains all leaves and all support vertices of G. In addition, if $x \notin D$, it joins two distinct components of $G[D]$, since G is acyclic. Hence

$$v_c^1(\times 2) \le c^1(\times 2) - 1.$$

(f) Suppose that $\mathcal{X} = e$. Then D contains all leaves of G by definition. Let $s \in S \setminus D$ be a support vertex of G that is not in D and let $L \cap N(s)$ be the set of leaves that are neighbors of s. Then there exists a component of $G[D]$ in distance less or equal $2k+1$ to $L \cap N(s)$. Using this observation and (i), we obtain

$$\begin{aligned} v_c^k(e) &\le 2k(|S| + |D \setminus (L \cup S)| - 1) \\ &= 2k(c^k(e) + |S \setminus D| - |L \setminus L_1| - 1). \end{aligned}$$

□

3.3 Relations between various domination parameters

Using inequality (5), the following results are immediate.

Corollary 3.3. *If G is a connected graph, then*

(a) (Meierling & Volkmann [11] 2005)

$$\gamma_c^k(G) \le (2k+1)\gamma^k(G) - 2k;$$

(b) (Hansberg, Meierling & Volkmann [7] 2007)

$$\gamma_c^k(G) \le \frac{3k+1}{2}\gamma_t^k(G) - 2k;$$

(c) $\gamma_c^k(G) \le 2k\gamma_{wc}^k(G) - (2k-1)$*;*

(d) $\gamma_c^k(G) \le (k+1)\gamma_{pr}^k(G) - 2k$*;*

(e) $\gamma_c(G) \le 2\gamma_{\times 2}(G) - 2$ *and* $\gamma_{p,c}(G) \le \frac{p+2}{p}\gamma_{\times p}(G) - 2$*;*

(f) $\gamma_{e,c}^k(G) \le (2k+1)\gamma_e^k(G) + 2k(|S| - |L|) - 2k.$

We would like to add the following remark.

Remark 3.4. *The inequalities in Corollary 3.3 (a) and (b) were shown for $k=1$ by Duchet and Meyniel [5] in 1982 and Favaron and Kratsch [6] in 1991, respectively.*

3.4 Lower bounds for various domination parameters in trees

In this section we focus on the class of trees. Using inequalities (7) and (8) combined with Corollary 3.3, we obtain lower bounds for several domination parameters.

Corollary 3.5. *If G is a tree on n vertices, L the set of leaves of G and S the set of support vertices of G, then*

(a) (Meierling & Volkmann [11] 2005; Raczek, Lemańska & Cyman [13] 2006)

$$\gamma^k(G) \geq \frac{n - k|L| + 2k}{2k+1};$$

(b) (Hansberg, Meierling & Volkmann [7] 2007)

$$\gamma_t^k(G) \geq \frac{2}{3k+1}(n - k|L| + 2k);$$

(c) $\gamma_{wc}^k(G) \geq \dfrac{n - k|L| + 2k - 1}{2k}$;

(d) $\gamma_{pr}^k(G) \geq \dfrac{n - k|L| + 2k}{k+1}$;

(e) (Chellali [2] 2006)

$$\gamma_{\times 2}(G) \geq \frac{2n + 2 - |S| + |L|}{3};$$

(f) $\gamma_e^k(G) \geq \dfrac{n + 2k - 2k(|S| - |L|)}{2k+1}$.

We would like to add the following remark.

Remark 3.6. *The inequality in Corollary 3.5 was shown for $k = 1$ by*

(a) Lemańska [9] in 2004;

(b) Chellali and Haynes [3] in 2006;

(c) Lemańska [10] in 2007;

(d) Raczek [12] in 2007.

The results presented in the next corollary follow from Corollary 3.5 and from the fact that every restrained as well as every weak dominating set is in particular an end-dominating set.

Corollary 3.7. *If G is a tree on n vertices, L is the set of leaves of G and S is the set of support vertices of G, then*

(a) (Chellali [1] 2005)

$$\gamma_r(G) \geq \frac{n+2+|L|-|S|}{3};$$

(b) (Domke, Hattingh, Henning & Markus [4] 2000)

$$\gamma_r(G) \geq \frac{n+2}{3};$$

(c) (Chellali [1] 2005)

$$\gamma_w(G) \geq \frac{n+2+|L|-|S|}{3};$$

(d) (Hattingh & Rautenbach [8] 2002)

$$\gamma_w(G) \geq \frac{n+2}{3};$$

(e) (Hattingh & Rautenbach [8] 2002)

$$\gamma_e(G) \geq \frac{n+2}{3}.$$

3.5 Characterization of the extremal trees

We begin with the definition of several classes of trees. A *star* is a complete bipartite graph $K_{1,q}$, where $q \geq 1$; the vertex in the partition set of size one is called the *center*. A *double star* consists of two stars whose centers are joined by an edge.

Definition 3.8. *Let $\mathcal{R}_1$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of stars whose edges have been $(k-1)$-times subdivided with centers $x_1, x_2, \ldots, x_r$. Join $x_1, x_2, \ldots, x_r$ by $r-1$ paths of length $2k+1$ such that the resulting graph is a tree.*

Let $\mathcal{R}_2$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of double stars whose edges are $(k-1)$-times subdivided with center pairs $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_r, y_r\}$. Join these pairs by $r-1$ paths of length $2k+1$ such that the resulting graph is a tree.

Let $\mathcal{R}_3$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of stars whose edges have been $(k-1)$-times subdivided with centers $x_1, x_2, \ldots, x_r$. Join $x_1, x_2, \ldots, x_r$ by $r-1$ paths of length $2k$ such that the resulting graph is a tree.

Let $\mathcal{R}_4$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of double stars whose end-edges are $(k-1)$-times subdivided with center pairs $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_r, y_r\}$. Join these pairs by $r-1$ paths of length $2k+1$ such that the resulting graph is a tree.

Let $\mathcal{R}_5$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of stars such that $|V(T_i)| \geq 3$ for $i = 1, 2, \ldots, r$ with centers $x_1, x_2, \ldots, x_r$ and let $P_1, P_2, \ldots, P_s$ be a set of paths of length one. Join $x_1, x_2, \ldots, x_r, P_1, P_2, \ldots, P_s$ by $r+s-1$ paths of length two such that the resulting graph is a tree.

Let $\mathcal{R}_6$ be the family of trees that is obtained as follows: Let $T_1, T_2, \ldots, T_r$ be a set of stars whose edges have been $2k$-times subdivided with centers $x_1, x_2, \ldots, x_r$. Join $x_1, x_2, \ldots, x_r$ by $r-1$ paths of length $2k+1$ such that the resulting graph is a tree.

Now we show that the above classes of trees fulfill some of the inequalities proved in the previous section with equality.

Theorem 3.9. *If G is a tree, then*

(a) (Meierling & Volkmann [11] 2005; Raczek, Lemańska & Cyman [13] 2006)

$$\gamma^k(G) = \frac{n - k|L| + 2k}{2k+1}$$

if and only if G belongs to the family $\mathcal{R}_1$;

(b) $\gamma_t^k(G) = \frac{2}{3k+1}(n - k|L| + 2k)$ *if and only if G belongs to the family $\mathcal{R}_2$;*

(c) $\gamma_{wc}^k(G) = \frac{n - k|L| + 2k - 1}{2k}$ *if and only if G belongs to the family $\mathcal{R}_3$;*

(d) $\gamma_{pr}^k(G) = \frac{n - k|L| + 2k}{k+1}$ *if and only if G belongs to the family $\mathcal{R}_4$;*

(e) (Chellali [2] 2006)

$$\gamma_{\times 2}(G) = \frac{2n + 2 - |S| + |L|}{3}$$

if and only if G belongs to the family $\mathcal{R}_5$;

(f) $\gamma_e^k(G) = \frac{n + 2k - 2k(|S| - |L|)}{2k+1}$ *if and only if G belongs to the family $\mathcal{R}_6$.*

Proof. It is easy to check that $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_6$ fulfill (7) or (8) with equality for the respective property.

If G is a tree that fulfills (7) with equality, then

$$\gamma_c^k(G) = n - kn_1, \tag{9}$$

where n_1 is the number of leaves of G. If G is a tree that fulfills (7) or (8) with equality, then

$$\gamma_c^k(G) = \gamma_{\mathcal{X}}^k(G) + v_c^k(\mathcal{X}). \tag{10}$$

Equality (9) implies that G is a tree whose end-edges have been $(k-1)$-times subdivided. The remaining equality (10) is fulfilled if and only if $c^k(\mathcal{X})$ and $v_c^k(\mathcal{X})$ are maximal which results in the classes $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_6$. □

We would like to add the following remark.

Remark 3.10. *The characterization in Theorem 3.9 was shown for $k = 1$ by*

(a) Lemańska [9] in 2004;

(b) Chellali and Haynes [3] in 2006;

(c) Lemańska [10] in 2007;

(d) Raczek [12] in 2007.

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