Total α-domination in graphs

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Abstract

Let $G = (V, E)$ be any graph without isolated vertices. For some α with $0 < \alpha \leq 1$ and a set $S \subseteq V$, we say that S is a total α-dominating set if for any $v \in V$, $|N(v) \cap S| \geq \alpha|N(v)|$. The cardinality of a smallest such set S is called the total α*-domination number* of G and is denoted by $\gamma_{t\alpha}(G)$. In this paper, we study total α-domination, and obtain several results and bounds for the total α-domination number of a graph G.

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1 Introduction

For notation and terminology in general we follow [4]. Let $G = (V, E) = (V(G), E(G))$ be a graph without isolated vertices. The *(open) neighborhood*

$N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v, and for a subset S, $N(S) = \bigcup_{v \in S} N(v)$. A set $S \subseteq V$ is said to *dominate* G if every vertex in G is either in S or is adjacent to some vertex in S. A set $S \subseteq V$ is said to *total-dominate* a graph if for any $v \in V(G)$, $|N(v) \cap S| \geq 1$. For $0 < \alpha \leq 1$, a set $S \subseteq V$ is said to *α-dominate* a graph G, if for any vertex $v \in V - S$, $|N(v) \cap S| \geq \alpha |N(v)|$. The minimum cardinality of a dominating set is the *domination number*, denoted $\gamma(G)$, the minimum cardinality of a total dominating set is the *total domination number*, denoted $\gamma_t(G)$, and the minimum cardinality of an α-dominating set is the *α-domination number*, denoted $\gamma_\alpha(G)$. We refer a dominating set of cardinality $\gamma(G)$ as a $\gamma(G)$-set, a total dominating set of cardinality $\gamma_t(G)$, as a $\gamma_t(G)$-set, and a α-dominating set of cardinality $\gamma_\alpha(G)$ as a $\gamma_\alpha(G)$-set. For references on α-domination in graphs see, for example, [1, 2, 3].

In this paper we initiate the study of *total α-domination* in graphs. For an α with $0 < \alpha \leq 1$ and a set $S \subseteq V$, we say that S is a total α-dominating set if for any $v \in V(G)$, $|N(v) \cap S| \geq \alpha |N(v)|$. The cardinality of a smallest such set S is called the *total α-domination number* and is denoted by $\gamma_{t\alpha}(G)$. We study some basic properties of total α-dominating sets and obtain several bounds and exact values for the total α-domination number of a graph G. We will refer a total α-dominating set of minimum cardinality as a $\gamma_{t\alpha}(G)$-set.

2 General results

First we determine the total α-domination number of the complete graph.

Proposition 1 *If K_n is a complete graph with n vertices, then $\gamma_{t\alpha}(K_n) = \lceil \alpha(n-1) \rceil + 1$.*

Proof. Let S be a $\gamma_{t\alpha}(K_n)$-set and let $x \in S$. So $|N(x) \cap S| \geq \lceil \alpha(n-1) \rceil$. But $|N(x) \cap S| = |S| - 1$. This implies that $|S| - 1 \geq \lceil \alpha(n-1) \rceil$, and so $|S| \geq \lceil \alpha(n-1) \rceil + 1$. Thus $\gamma_{t\alpha}(K_n) \geq \lceil \alpha(n-1) \rceil + 1$. On the other hand any subset of vertices of cardinality $\lceil \alpha(n-1) \rceil + 1$ is a total α-dominating set. This completes the proof. ■

Theorem 2 *If $G = K_{n_1, n_2, \ldots, n_p}$ is the complete p-partite graph of order n with $p \geq 2$, then*

$$\frac{1}{p-1} \sum_{i=1}^{p} \lceil \alpha(n - n_i) \rceil \leq \gamma_{t\alpha}(G) \leq \sum_{i=1}^{p} \lceil \alpha n_i \rceil.$$

Proof. Assume that $V_1, V_2, \ldots, V_p$ are the partite sets of the complete p-partite graph $G = K_{n_1,n_2,\ldots,n_p}$ such that $|V_i| = n_i$ for $1 \leq i \leq p$.

Let S be a $\gamma_{t\alpha}(G)$-set. If $x_i \in V_i$, then

$$|(V(G) - V_i) \cap S| = |N(x_i) \cap S| \geq \alpha |N(x_i)| = \alpha(n - n_i)$$

and thus

$$|(V(G) - V_i) \cap S| \geq \lceil \alpha(n - n_i) \rceil$$

for each $1 \leq i \leq p$. This implies that

$$\begin{aligned} (p-1)\gamma_{t\alpha}(G) &= (p-1)|S| = (p-1)|V(G) \cap S| \\ &= \sum_{i=1}^{p} |(V(G) - V_i) \cap S| \geq \sum_{i=1}^{p} \lceil \alpha(n - n_i) \rceil, \end{aligned}$$

and the first inequality is proved.

Now let $V_i' \subseteq V_i$ be a subset of cardinality $\lceil \alpha n_i \rceil$ for $1 \leq i \leq p$. If we define $S = \bigcup_{i=1}^{p} V_i'$, then we will show that S is a total α-dominating set of G. If x is an arbitrary vertex of G, then $x \in V_j$ for a fixed index $1 \leq j \leq p$. It follows that

$$\begin{aligned} |N(x) \cap S| &= \sum_{i=1}^{p} |V_i'| - |V_j'| = \sum_{i=1}^{p} \lceil \alpha n_i \rceil - \lceil \alpha n_j \rceil \\ &\geq \lceil \alpha(n - n_j) \rceil = \lceil \alpha |N(x)| \rceil \geq \alpha |N(x)|. \end{aligned}$$

Therefore S is a total α-dominating set of G of cardinality $\sum_{i=1}^{p} \lceil \alpha n_i \rceil$, and the second inequality is proved. ■

The following example will demonstrate that both of the inequalities in Theorem 2 are sharp.

Let $H = K_{n_1,n_2,\ldots,n_p}$ be the complete p-partite graph with $p \geq 2$ such that $n_1 = n_2 = \ldots = n_p$, and choose $\alpha = 1/n_1$. In this case we obtain

$$\begin{aligned} \frac{1}{p-1} \sum_{i=1}^{p} \lceil \alpha(n - n_i) \rceil &= \frac{1}{p-1} \sum_{i=1}^{p} \lceil \alpha(p-1) n_1 \rceil \\ &= \frac{(p-1)p}{p-1} = p = \sum_{i=1}^{p} \lceil \alpha n_i \rceil. \end{aligned}$$

Consequently, Theorem 2 implies that

$$\gamma_{t\alpha}(H) = p = \frac{1}{p-1}\sum_{i=1}^{p}\lceil \alpha(n-n_i)\rceil = \sum_{i=1}^{p}\lceil \alpha n_i\rceil.$$

The special case $p = 2$ in Theorem 2 leads to the next result immediately.

Corollary 3 *If K_{n_1,n_2} is the complete bipartite graph, then*

$$\gamma_{t\alpha}(K_{n_1,n_2}) = \lceil \alpha n_1\rceil + \lceil \alpha n_2\rceil.$$

We now study some basic properties of the total α-domination number of graphs.

Proposition 4 *For any graph G of order n,*
(1) $\gamma(G) \le \gamma_\alpha(G) \le \gamma_{t\alpha}(G) \le n$.
(2) $\gamma_t(G) \le \gamma_{t\alpha}(G)$.
(3) If $\alpha_1 \le \alpha_2$, then $\gamma_{t\alpha_1}(G) \le \gamma_{t\alpha_2}(G)$.
(4) The difference $\gamma_{t\alpha}(G) - \gamma_t(G)$ in a graph G can be arbitrarily large.
(5) The difference $\gamma_{t\alpha}(G) - \gamma_\alpha(G)$ in a graph G can be arbitrarily large.
(6) $\gamma_{t1}(G) = n$.

Proof. The inequalities (1)-(3) are obvious. Property (4) follows from Proposition 1. Property (5) follows from Corollary 3 with $n_1 = 1$. To prove (6) first notice that $\gamma_{t\alpha}(G) \le n$. Let S be a $\gamma_{t1}(G)$-set and $v \in V(G)$. Then $|N(v) \cap S| \ge \alpha|N(v)| = |N(v)|$. This implies that S contains all neighbors of any vertex of G, and so $|S| \ge n$. ■

Thus henceforth we consider only $\alpha < 1$.

Proposition 5 *(1) If $\alpha > 1 - \frac{1}{\Delta(G)}$, then $\gamma_{t\alpha}(G) = n$.*
(2) If $0 < \alpha \le \frac{1}{\Delta(G)}$, then $\gamma_{t\alpha}(G) = \gamma_t(G)$.

Proof. (1) It is obvious that $\gamma_{t\alpha}(G) \le n$. Let S be a $\gamma_{t\alpha}(G)$-set. Suppose to the contrary that $|S| < n$. Let $x \notin S$ and $y \in N(x) \cap S$. It follows that $deg(y) - 1 \ge |N(y) \cap S| \ge \alpha deg(y)$. This implies that

$$\alpha \le 1 - \frac{1}{deg(y)} \le 1 - \frac{1}{\Delta(G)}.$$

This is a contradiction and thus $|S| = n$.

(2) First notice that $\gamma_{t\alpha}(G) \geq \gamma_t(G)$, since any $\gamma_{t\alpha}(G)$-set is also a total dominating set. Let S be a $\gamma_t(G)$-set. For any $v \in V(G)$, $|N(v) \cap S| \geq 1$. By assumption we obtain $|N(v) \cap S| \geq 1 \geq \alpha\Delta(G) \geq \alpha|N(v)|$. This implies that S is a total α-dominating set, and so the result follows. ■

If P_n and C_n are the path and the cycle of order n, then by [5] we have the following.

$$\gamma_t(P_n) = \gamma_t(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Using these identities and Proposition 5, we obtain the next result immediately.

Corollary 6 *(1)*

$$\gamma_{t\alpha}(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor & \text{if } 0 < \alpha \leq \frac{1}{2} \\ n & \text{if } \frac{1}{2} < \alpha \leq 1 \end{cases}.$$

(2)

$$\gamma_{t\alpha}(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor & \text{if } 0 < \alpha \leq \frac{1}{2} \\ n & \text{if } \frac{1}{2} < \alpha \leq 1 \end{cases}.$$

We next characterize graphs G with $\gamma_{t\alpha}(G) \in \{2, n\}$. For $\alpha \in (0, 1)$, let $\mathcal{F}_\alpha^2$ be the class of all graphs G containing two adjacent vertices v_1, v_2 each of degree at most $\frac{1}{\alpha}$ such that $V(G) = N(v_1) \cup N(v_2)$, each vertex in $N(v_i) \backslash N(v_j)$ is of degree at most $\frac{1}{\alpha}$ for $i \neq j$, and each vertex in $N(v_1) \cap N(v_2)$ is of degree at most $\frac{2}{\alpha}$.

Theorem 7 *Let G be a graph and $\alpha \in (0, 1)$. Then $\gamma_{t\alpha}(G) = 2$ if and only if $G \in \mathcal{F}_\alpha^2$.*

Proof. ($\Longrightarrow$) If $G \in \mathcal{F}_\alpha^2$, then G contains two adjacent vertices v_1, v_2 with above properties. Clearly $\{v_1, v_2\}$ is a total α-dominating set, and so $\gamma_{t\alpha}(G) \leq 2$. Then from Proposition 4 part (2) we obtain $\gamma_{t\alpha(G)} = 2$.

($\Longleftarrow$) Assume that $\gamma_{t\alpha}(G) = 2$. Let $S = \{x, y\}$ be a $\gamma_{t\alpha}(G)$-set. Clearly x is adjacent to y. Furthermore, $1 = |N(x) \cap S| \geq \alpha deg(x)$ and $1 = |N(y) \cap S| \geq \alpha deg(y)$. These imply that x, y are of degree at most $\frac{1}{\alpha}$. Let $z \in V(G) \setminus S$. If $z \in N(x) \cap N(y)$, then $2 = |N(z) \cap S| \geq \alpha deg(z)$, which implies that $deg(z) \leq \frac{2}{\alpha}$. It remains to assume that $z \notin N(x) \cap N(y)$. Now $1 = |N(z) \cap S| \geq \alpha deg(z)$, and so $deg(z) \leq \frac{1}{\alpha}$. We conclude that $G \in \mathcal{F}_{\alpha}^{2}$. ■

We recall that in a graph G, $S(G)$ is the set of all support vertices of G.

Theorem 8 *Let G be a graph G of order n and $\alpha \in (0,1)$. If G is not a star, then $\gamma_{t\alpha}(G) = n$ if and only if for each vertex $x \in V(G) \setminus S(G)$, there is a vertex $y \in N(x)$ such that $deg(y) < \frac{1}{1-\alpha}$.*

Proof. ($\Longrightarrow$) Assume that $\gamma_{t\alpha}(G) = n$. Let $x \in V(G) \setminus S(G)$ be an arbitrary vertex. Suppose to the contrary that for all vertices $y \in N(x)$, $deg(y) \geq \frac{1}{1-\alpha}$. Let $S = V(G) \setminus \{x\}$. We show that S is a total α-dominating set of G. First notice that $|N(x) \cap S| = deg(x) \geq \alpha deg(x)$. If $y \in N(x)$, then $|N(y) \cap S| = deg(y) - 1 \geq \alpha deg(y)$, since $deg(y) \geq \frac{1}{1-\alpha}$. Finally, assume that $v \notin N(x) \cup \{x\}$. Then $|N(v) \cup S| = deg(v) \geq \alpha deg(v)$. Thus S is a total α-dominating set of G, a contradiction to the assumption $\gamma_{t\alpha}(G) = n$.

($\Longleftarrow$) Assume to the contrary that $\gamma_{t\alpha}(G) < n$. Let S be a $\gamma_{t\alpha}(G)$-set, and let $x \notin S$. Clearly $x \notin S(G)$. The hypothesis implies that there is a vertex $y \in N(x)$ such that $deg(y) < \frac{1}{1-\alpha}$. Now $|N(y) \cap S| \geq \alpha deg(y)$. But $|N(y) \cap S| \leq deg(y) - 1$, since $x \notin S$. Thus we obtain $deg(y) - 1 \geq \alpha deg(y)$. This implies that $deg(y) \geq \frac{1}{1-\alpha}$, a contradiction. Thus, $\gamma_{t\alpha}(G) = n$. ■

The wheel W_4 of order four is isomorphic to the complete graph K_4 whose total α-domination number has already been determined. For the wheel W_5 of order five it is easy to check that

$$\gamma_{t\alpha}(W_5) = \begin{cases} 5 & \text{if } \alpha > \frac{2}{3} \\ 3 & \text{if } \frac{1}{3} < \alpha \leq \frac{2}{3} \\ 2 & \text{if } \alpha \leq \frac{1}{3} \end{cases}.$$

Theorem 9 *If W_n is a wheel of order $n \geq 6$ and $0 < \alpha < 1$, then*

$$\gamma_{t\alpha}(W_n) = \begin{cases} n & \textit{if } \alpha > \frac{2}{3} \\ 1 + \gamma_t(C_{n-1}) & \textit{if } \frac{1}{3} < \alpha \leq \frac{1}{2} + \varepsilon(n) \\ 1 + \lceil \alpha(n-1) \rceil & \textit{else} \end{cases},$$

where $\varepsilon(n)$ is defined as

$$\varepsilon(n) = \begin{cases} \frac{1}{2(n-1)} & \textit{if } n = 4r \\ 0 & \textit{if } n = 4r+1 \\ \frac{3}{2(n-1)} & \textit{if } n = 4r+2 \\ \frac{1}{n-1} & \textit{if } n = 4r+3 \end{cases}.$$

Proof. Let S be a $\gamma_{\alpha t}$-set of W_n. Let x be the unique vertex of degree $n-1$. By $v_1 v_2 \cdots v_{n-1} v_1$ we denote $W_n - x$, a cycle of order $n-1$.

If $\alpha > \frac{2}{3}$, then we have $|N(v_i) \cap S| \geq 3\alpha > 2$ for $1 \leq i \leq n-1$. Hence $\bigcup_{i=1}^{n-1} N(v_i) \subset S$ and thus, $S = V(W_n)$.

If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, let $t = \max\{\gamma_t(C_{n-1}), \lceil \alpha(n-1) \rceil\}$. The set $\{x\} \cup S'$, where S' is a total dominating set of $W_n - x$ of order $\alpha(n-1)$, is a total α-dominating set of W_n. It follows that $\gamma_{t\alpha}(W_n) \leq 1 + t$. On the other hand, we have $|N(v_i) \cap S| \geq 3\alpha > 1$ for $1 \leq i \leq n-1$. If $x \notin S$, then $\bigcup_{i=1}^{n-1} N(v_i) \subset S \cup \{x\}$ and thus, $S = V(W_n) - \{x\}$, contradiction to the choice of S. So $x \in S$. Since $|N(v_i) \cap S| > 1$ for $1 \leq i \leq n-1$, the set $S - \{x\}$ is a total dominating set of $W_n - x$. Moreover, $|N(x) \cap S| \geq \alpha(n-1)$ and thus, $\gamma_{t\alpha}(W_n) \geq 1 + t$.

Under the assumption that $\frac{1}{3} < \alpha \leq \frac{2}{3}$, we have

$$\gamma_t(C_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n-1}{4} \right\rceil - \left\lfloor \frac{n-1}{4} \right\rfloor < \lceil \alpha(n-1) \rceil$$

if and only if $n = 4r$ and $\alpha > \frac{2r}{4r-1}$, or $n = 4r+1$ and $\alpha > \frac{1}{2}$, or $n = 4r+2$ and $\alpha > \frac{2r+2}{4r+1}$, or $n = 4r+3$ and $\alpha > \frac{2r+1}{4r+2}$.

If $\alpha \leq \frac{1}{3}$, then $\{x\} \cup T$, where T is any subset of $\{v_1, v_2, \ldots, v_{n-1}\}$ of order $\lceil \alpha(n-1) \rceil$, is a total α-dominating set of W_n. It follows that $\gamma_{t\alpha}(W_n) \leq 1 + \lceil \alpha(n-1) \rceil$. On the other hand, we have $|N(v_i) \cap S| \geq 1$. If $x \notin S$, then S is a total dominating set of $W_n - x$. Hence $|S| \geq \gamma_t(C_{n-1}) = \lfloor \frac{n-1}{2} \rfloor + \lceil \frac{n-1}{4} \rceil - \lfloor \frac{n-1}{4} \rfloor$, a contradiction to the choice of S for $n = 7$ and $n \geq 10$. For $n = 6, 8, 9$ it follows that $|S| \geq 1 + \lceil \alpha(n-1) \rceil$. So assume that $x \in S$. In that case $\gamma_{t\alpha}(W_n) \geq 1 + |N(x) \cap S| \geq 1 + \lceil \alpha(n-1) \rceil$. ■

The fan F_3 of order three is isomorphic to the complete graph K_3 whose total α-domination number has already been determined. For the fans F_n of order $n = 4, 5$, respectively, it is easy to check that

$$\gamma_{t\alpha}(F_n) = \begin{cases} n & \text{if } \alpha > \frac{2}{3} \\ 3 & \text{if } \frac{1}{3} < \alpha \le \frac{2}{3} \\ 2 & \text{if } \alpha \le \frac{1}{3} \end{cases} .$$

Theorem 10 *If F_n is a fan of order $n \ge 6$ and $0 < \alpha < 1$, then*

$$\gamma_{t\alpha}(F_n) = \begin{cases} n & \text{if } \alpha > \frac{2}{3} \\ 1 + \gamma_t(P_{n-3}) & \text{if } \frac{1}{3} < \alpha \le \frac{1}{2} - \varepsilon(n) \\ 1 + \lceil \alpha(n-1) \rceil & \text{else} \end{cases} ,$$

where $\varepsilon(n)$ is defined as

$$\varepsilon(n) = \begin{cases} \frac{1}{2(n-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n = 4r+1 \\ \frac{1}{n-1} & \text{if } n = 4r+3 \end{cases} .$$

Proof. Let S be a $\gamma_{\alpha t}$-set of F_n with the property that it contains as few vertices of degree two as possible. Let x be the unique vertex of degree $n-1$. By $v_1 v_2 \cdots v_{n-1}$ we denote $F_n - x$, a path of order $n-1$.

If $\alpha > \frac{2}{3}$, then for $2 \le i \le n-2$ we have $|N(v_i) \cap S| \ge 3\alpha > 2$. Hence $\bigcup_{i=2}^{n-2} N(v_i) \subset S$ and thus, $S = V(F_n)$.

If $\frac{1}{2} < \alpha \le \frac{2}{3}$, then $\alpha(n-1) > \frac{1}{2}(n-1)$ and therefore $\lceil \alpha(n-1) \rceil \ge \lfloor \frac{n-1}{2} \rfloor + \lceil \frac{n-1}{4} \rceil - \lfloor \frac{n-1}{4} \rfloor = \gamma_t(P_{n-1})$. Hence $\{x\} \cup S'$, where S' is a total dominating set of $F_n - x$ of order $\lceil \alpha(n-1) \rceil$, is a total α-dominating set of F_n. It follows that $\gamma_{t\alpha}(F_n) \le 1 + \lceil \alpha(n-1) \rceil$. On the other hand, we have $|N(v_1) \cap S| \ge 2\alpha > 1$ which implies that $x \in S$. Moreover, $|N(x) \cap S| \ge \alpha(n-1)$ and thus, $|S| = \gamma_{t\alpha}(F_n) \ge 1 + \lceil \alpha(n-1) \rceil$.

If $\frac{1}{3} < \alpha \le \frac{1}{2}$, then let $t = \max\{\gamma_t(P_{n-3}), \lceil \alpha(n-1) \rceil\}$. The set $\{x\} \cup S'$, where S' is a total dominating set of $F_n - \{x, v_1, v_{n-1}\}$ of order t, is a total α-dominating set of F_n. It follows that $\gamma_{t\alpha}(F_n) \le 1 + t$. On the other hand, we have $|N(v_i) \cap S| \ge 3\alpha > 1$. If $x \notin S$, then $\bigcup_{i=2}^{n-2} N(v_i) \subset S \cup \{x\}$ and thus, $S = V(F_n) - \{x\}$, a contradiction to the choice of $|S|$. So $x \in S$. Note that by the choice of S neither v_1 nor v_{n-1} is in S. Since $|N(v_i) \cap S| > 1$ and $|N(x) \cap S| \ge \alpha(n-1)$, the set $S - \{x\}$ is a total dominating set of $F_n - x$ of order at least $\lceil \alpha(n-1)\rceil$. This proves $\gamma_{t\alpha}(F_n) \geq 1+t$. Under the assumption that $\frac{1}{3} < \alpha \leq \frac{1}{2}$, we have

$$\gamma_t(P_{n-3}) = \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lceil \frac{n-3}{4} \right\rceil - \left\lfloor \frac{n-3}{4} \right\rfloor < \lceil \alpha(n-1)\rceil$$

if and only if $n = 4r$ and $\alpha > \frac{2r-1}{4r-1}$, or $n = 4r+2$ and $\alpha > \frac{2r}{4r+1}$, or $n = 4r+3$ and $\alpha > \frac{2r}{4r+2}$.

If $\alpha \leq \frac{1}{3}$, then $\{x\} \cup T$, where T is any subset of $\{v_1, v_2, \ldots, v_{n-1}\}$ of order $\lceil \alpha(n-1)\rceil$, is a total α-dominating set of F_n. It follows that $\gamma_{t\alpha}(F_n) \leq 1 + \lceil \alpha(n-1)\rceil$. On the other hand, we have $|N(x) \cap S| \geq \lceil \alpha(n-1)\rceil$. If $x \notin S$, then S is a total dominating set of $F_n - x$. Hence $|S| \geq \gamma_t(P_{n-1})$, a contradiction to the choice of S for $n = 7$ and $n \geq 10$. For $n = 6, 8, 9$ it follows that $|S| \geq 1 + \lceil \alpha(n-1)\rceil$. So assume that $x \in S$. In that case $\gamma_{t\alpha}(F_n) \geq 1 + \lceil \alpha(n-1)\rceil$. ■

3 Bounds

Theorem 11 *For any graph G with n vertices, $\gamma_{t\alpha}(G) \geq \lceil \frac{n\alpha\delta(G)}{\Delta(G)} \rceil$.*

Proof. Let S be a $\gamma_{t\alpha}(G)$-set. Let M be the set of all edges between S and $V(G) - S$, and let N be the set of all edges in $G[S]$. It follows that

$$\sum_{v\in S} deg(v) = 2N + M. \tag{1}$$

For any vertex $v \in V(G)$, $|N(v) \bigcap S| \geq \alpha|N(v)|$. Summing over $V(G)$ yields

$$\sum_{v\in V(G)} |N(v) \cap S| \geq \sum_{v\in V(G)} \alpha|N(v)|$$

and so

$$\sum_{v\in S} |N(v) \cap S| + \sum_{v\in V(G)\setminus S} |N(v) \cap S| \geq \sum_{v\in V(G)} \alpha|N(v)|.$$

But $\sum_{v\in S} |N(v) \cap S| + \sum_{v\in V(G)\setminus S} |N(v) \cap S| = 2N + M$. So

$$2N + M \geq \sum_{v\in V(G)} \alpha|N(v)| = \sum_{v\in V(G)} \alpha deg(v),$$

and by (1)

$$\sum_{v\in S} deg(v) \geq \sum_{v\in V(G)} \alpha deg(v). \tag{2}$$

Now

$$\begin{aligned} |S|\Delta(G) &\geq \sum_{v\in S} deg(v) \\ &\geq \sum_{v\in V(G)} \alpha deg(v) \\ &\geq n\alpha\delta(G) \end{aligned}$$

and thus $|S| \geq \frac{n\alpha\delta(G)}{\Delta(G)}$. ∎

Corollary 12 *For any tree T of order n, $\gamma_{t\alpha}(T) \geq \frac{n\alpha}{\Delta(T)}$.*

Corollary 13 *If G is a k-regular graph G of order n and $0 < \alpha \leq 1$, then $\gamma_{t\alpha}(G) \geq \lceil n\alpha \rceil$.*

Theorem 14 *If a graph G has m edges, then $\gamma_{t\alpha}(G) \geq \lceil \frac{2m\alpha}{\Delta(G)} \rceil$.*

Proof. We follow the proof of Theorem 11. Let S be a $\gamma_{t\alpha}(G)$-set. From (2) we obtain

$$|S|\Delta(G) \geq \sum_{v\in S} \deg(v) \geq \sum_{v\in V} \alpha \deg(v) = \alpha 2m.$$

This implies that $\gamma_{t\alpha}(G) \geq \frac{2m\alpha}{\Delta(G)}$. ∎

The example after Theorem 2 demonstrates that Corollary 13 as well as Theorem 14 are sharp.

Theorem 15 *If a graph G has m edges, then $\gamma_{t\alpha}(G) \geq \lceil \frac{2(n\alpha\delta(G)-m)}{\alpha\delta(G)} \rceil$.*

Proof. Let S be a $\gamma_{t\alpha}(G)$-set. Let E_1 be the set of all edges between S and $G - S$. It follows that

$$\begin{aligned}|E(G)| &\geq |E(G[S])| + |E_1| \\ &\geq \frac{1}{2}\sum_{v\in S}\alpha deg(v) + \sum_{v\in V(G)\setminus S}|N(v)\cap S| \\ &\geq \frac{\alpha\delta(G)\gamma_{t\alpha}(G)}{2} + \alpha\delta(G)(n - \gamma_{t\alpha}(G)) \\ &= \frac{2n\alpha\delta(G) - \alpha\delta(G)\gamma_{t\alpha}(G)}{2}.\end{aligned}$$

This completes the proof. ■

The following is along with a similar result for α-domination in [1].

Proposition 16 *For $\frac{i}{i+1} < \alpha \leq \frac{i+1}{i+2}$ and $\epsilon > 0$, there is a connected graph G of order n such that $\gamma_{t\alpha}(G) \geq (\frac{i+1}{i+2} - \epsilon)n$.*

Proof. For $l \geq 1$, let $G(l)$ be a graph obtained from disjoint union of l cliques $H_1, H_2, ..., H_l$ of order $i + 2$ and one isolated vertex v by joining v and one vertex u_j of H_j by a new edge for $1 \leq j \leq l$. Let D be a minimum total α-dominating set of $G(l)$. If $V(H_j)\setminus\{u_j\} \subseteq D$ for some $1 \leq j \leq l$, then $|D \cap V(H_j)| \geq i + 1$. If $V(H_j) \setminus \{u_j\} \nsubseteq D$ for some $1 \leq j \leq l$, then there is a vertex $u'_j \in V(H_j) \setminus (\{u_j\} \cup D)$ with $d_{G(l)}(u'_j) = i + 1$ and thus

$$|D \cap V_{H_j}| \geq \lceil \alpha d_{G(l)}(u'_j)\rceil = \lceil \alpha(i + 1)\rceil = i + 1.$$

Thus $|D\cap V(H_j)| \geq i+1$ for $j = 1, 2, ..., l$, and so $|D| \geq l(i+1)$. This implies that $\gamma_{t\alpha}(G(l)) \geq l(i + 1)$ and thus $\frac{\gamma_{t\alpha}(G(l))}{n_{G(l)}} \geq \frac{l(i+1)}{l(i+2)+1}$. Since

$$\lim_{l\to\infty}\frac{l(i + 1)}{l(i + 2) + 1} = \frac{i + 1}{i + 2},$$

the desired bound follows. ■

In the rest of this section we study the affection of total α-domination number under vertex removal, edge removal and edge addition. We recall that a *pendant* edge in a graph G is an edge which is incident to an end-vertex.

Theorem 17 *If $e \in E(G)$ is a non-pendant edge of the graph G, then*

$$\gamma_{t\alpha}(G) - 2 \leq \gamma_{t\alpha}(G - e) \leq \gamma_{t\alpha}(G) + 2.$$

These bounds are sharp.

Proof. Let $e = xy \in E(G)$. We first establish the right inequality. Let S be a $\gamma_{t\alpha}(G)$-set. If $\{x, y\} \cap S = \emptyset$, then S is a total α-dominating set for $G - e$. So we assume that $\{x, y\} \cap S \neq \emptyset$. Let $y \in S$. Assume that $x \notin S$. If $N(x) \subseteq S$, then since e is not a pendant edge, we have $N_{G-e}(x) \subseteq S$, and so $|N_{G-e}(x) \cap S| = deg_{G-e}(x) \geq \alpha deg_{G-e}(x)$. Thus we assume that $N(x) \nsubseteq S$. Let $w \in N(x) \setminus S$. Then $S \cup \{w\}$ is a total α-dominating set for $G - e$. Next assume that $x \in S$. We proceed with Fact 1.

Fact 1. If $N(x) \subseteq S$, then $\gamma_{t\alpha}(G - e) \leq \gamma_{t\alpha}(G) + 1$.

Proof. Assume that $N(x) \subseteq S$. Since e is not a pendant edge, $N_{G-e}(x) \subseteq S$, and so $|N_{G-e}(x) \cap S| = deg_{G-e}(x) \geq \alpha deg_{G-e}(x)$. If $N(y) \subseteq S$, then similarly $|N_{G-e}(y) \cap S| \geq \alpha deg_{G-e}(y)$, and thus S is a total α-dominating set for $G - e$. It remains to assume that $N(y) \nsubseteq S$. Let $y_1 \in N(y) \setminus S$. Then $S \cup \{y_1\}$ is a total α-dominating set for $G - e$. We deduce that $\gamma_{t\alpha}(G - e) \leq \gamma_{t\alpha}(G) + 1$. $\diamond$

Similar to Fact 1, we obtain that if $N(y) \subseteq S$, then $\gamma_{t\alpha}(G - e) \leq \gamma_{t\alpha}(G) + 1$. Thus for next we assume that $N(x) \nsubseteq S$ and $N(y) \nsubseteq S$. Then $S \cup \{w_1, w_2\}$ is a total α-dominating set for G, where $w_1 \in N(x) - S$ and $w_2 \in N(y) - S$. We deduce that $\gamma_{t\alpha}(G - e) \leq \gamma_{t\alpha}(G) + 2$.

We next establish the left inequality. Let D be a $\gamma_{t\alpha}(G - e)$-set. If $\{x, y\} \subseteq D$, then D is a total α-dominating set for G. So assume that $x \notin D$. If $y \in D$, then $D \cup \{x\}$ is a total α-dominating set for G. We thus assume that $y \notin D$. Then $D \cup \{x, y\}$ is a total α-dominating set for G. We conclude that $\gamma_{t\alpha}(G) \leq \gamma_{t\alpha}(G - e) + 2$ which yields the left inequality.

To see the sharpness of left inequality let G_1, G_2 be two copies of P_8, and let x be one of the two central vertices of G_1 and y be one of the two central vertices of G_2. Let G be a graph obtained from G_1, G_2 by adding new edge $e = xy$. It follows that $\gamma_{t(\frac{1}{2})}(G - e) = 8$ and $\gamma_{t(\frac{1}{2})}(G) = 10 = \gamma_{t(\frac{1}{2})}(G - e) + 2$. For the right inequality let e be the central edge of P_4. We observe that $\gamma_{t(\frac{1}{2})}(P_4) = 2$ and $\gamma_{t(\frac{1}{2})}(P_4 - e) = 4 = \gamma_{t(\frac{1}{2})}(P_4) + 2$. ■

Corollary 18 *If $e \notin E(G)$, then $\gamma_{t\alpha}(G) - 2 \leq \gamma_{t\alpha}(G + e) \leq \gamma_{t\alpha}(G) + 2$ and these bounds are sharp.*

We recall that $S(G)$ is the set of all support vertices of G.

Theorem 19 *For any vertex* $v \in V(G) \setminus S(G)$,

$$\gamma_{t\alpha}(G) - \alpha deg(v) - 1 \leq \gamma_{t\alpha}(G - v) \leq \gamma_{t\alpha}(G) + deg(v) - 1.$$

These bounds are sharp.

Proof. Let $v \in V(G) \setminus S(G)$. Let S be a $\gamma_{t\alpha}(G) - set$. If $v \notin S$, then S is a total α-dominating set for $G - v$. Assume that $v \in S$. If $S \setminus \{v\}$ is not a total α-dominating set for $G - v$, then there is a vertex $u \in N(v)$ such that $|N_{G-v}(u) \cap (S \setminus \{v\})| < \alpha deg_{G-v}(u)$. Let $A = \{u \in N(v) : |N_{G-v}(u) \cap (S \setminus \{v\})| < \alpha deg_{G-v}(u)\}$. It is obvious that $(A \setminus S) \cap S(G) = \emptyset$. For each $u \in A$, we choose a vertex $u^* \in N_{G-v}(u)$. Let $B = \{u^* : u \in A\}$. Then $(S \setminus \{v\}) \cup B$ is a total α-dominating set for $G - v$. So $\gamma_{t\alpha}(G - v) \leq \gamma_{t\alpha}(G) + deg(v) - 1$.

Now let D be a $\gamma_{t\alpha}(G - v)$-set. It follows that $D \cup \{v\} \cup D_1$ is a total α-dominating set for G, where $D_1 \subseteq N(v)$ and $|D_1| = \lceil \alpha deg(v) \rceil$. This implies that $\gamma_{t\alpha}(G) \leq \gamma_{t\alpha}(G - v) + \alpha deg(v) + 1$.

To see the sharpness of left inequality by Corollary 6 $\gamma_{t(\frac{1}{2})}(P_8) = 4$. If G is a graph obtained from P_8 by adding a new vertex v and joining v to the two central vertices of P_8, then we observe that $\gamma_{t(\frac{1}{2})}(G) = 6 = \gamma_{t(\frac{1}{2})}(G - v) + \frac{1}{2} deg(v) + 1$. For the right inequality, let v be the central vertex of $K_{1,3}$. We subdivide any edge of $K_{1,3}$, and then join v to a leaf of the resulted graph to obtain a graph G. It follows that $\gamma_{t(\frac{1}{2})}(G) = 3$, while $\gamma_{t(\frac{1}{2})}(G - v) = 6 = \gamma_{t(\frac{1}{2})}(G) + deg(v) - 1$. ■

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